ECE 307 – Techniques for Engineering Decisions

Lecture 6. Transshipment and Shortest Path Problems

George Gross

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

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TRANSSHIPMENT PROBLEMS

- **We consider the shipment of a *homogeneous* commodity or product from a specified point or *source* to a particular destination or *sink*: the homogeneity characteristic ensures that each unit shipped is identical and is independent of point of origin**
- **Typically, the *source* and the *sink* are not directly connected; rather, the flow goes through the *transshipment points*, i.e., the intermediate nodes**
- **The objective is to determine the *maximal flow* from the *source* to the *sink***

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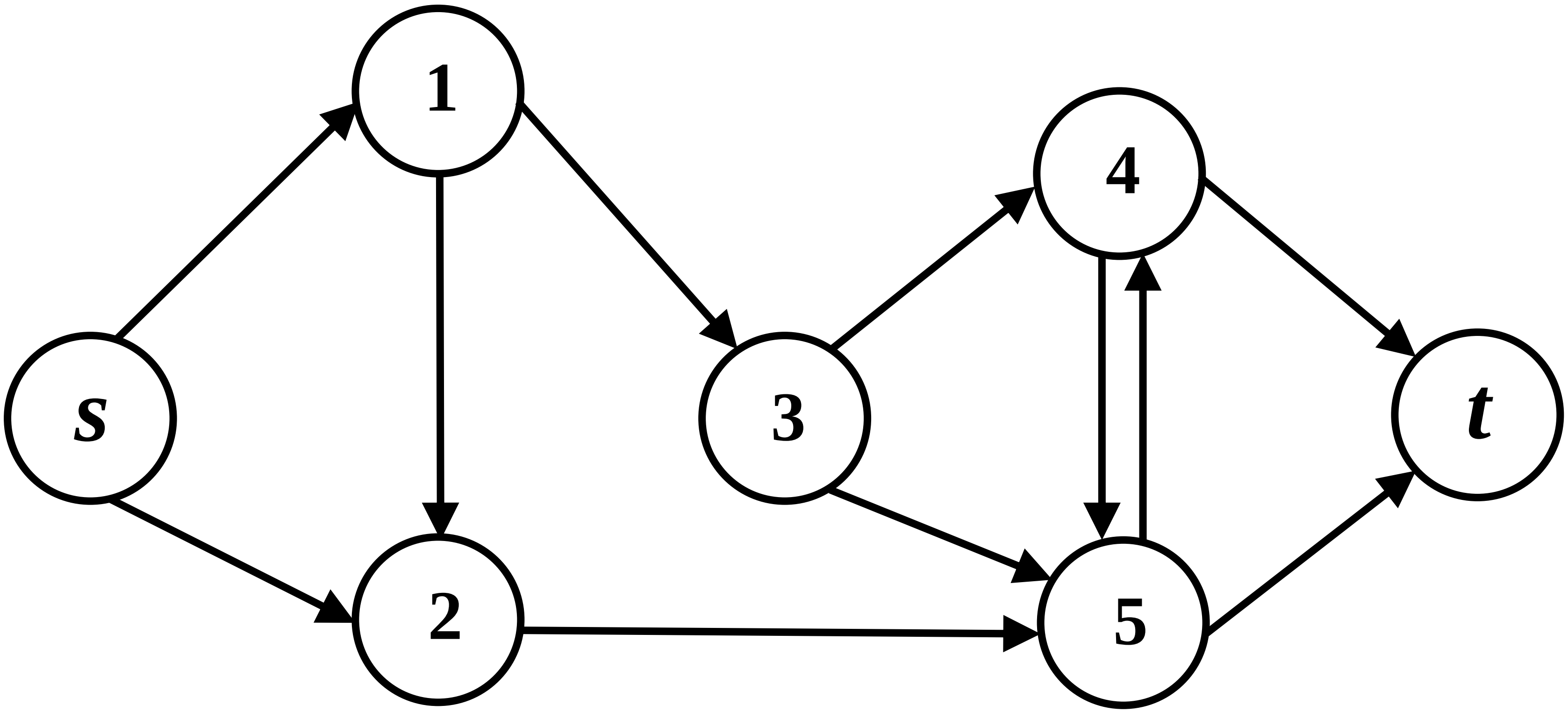
DIRECTED FLOW NETWORK EXAMPLE



TRANSSHIPMENT PROBLEMS

- **nodes 1, 2, 3, 4 and 5 are the *transshipment points***
- ***directed arcs* of the network are (*s*, 1), (*s*, 2), (1, 2), (1, 3), (2, 5), (3, 4), (3, 5), (4, 5), (5, 4), (4, *t*), (5, *t*) ; the existence of an arc from 4 to 5 and from 5 to 4 allows bi-directional flows between the two nodes**
- **each arc may be constrained in terms of a *limit on the flow through the arc***

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MAX FLOW PROBLEM

- We denote by f_{ij} the flow from i to j, which equals the amount of the commodity shipped from i to j on the arc (i, j) that directly connects the node i to the node j
- The problem is to determine the maximal flow f from s to t taking into account the *flow limits* k_{ij} of each arc (i, j)
- The mathematical statement of the problem is

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MAX FLOW PROBLEM

$$\max \quad Z = f$$

s.t.

$$0 \leq f_{ij} \leq k_{ij}$$ $\forall$ *arc* (i, j) *that connects nodes i and j*

$$f = \sum_i f_{si}$$ *at source s*

$$\sum_i f_{it} = f$$ *at sink t*

} *conservation of flow relations*

$$\sum_i f_{ij} = \sum_k f_{jk}$$ } *at each transshipment node j*

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MAX FLOW PROBLEM

- While we may use the simplex approach to solve the *max flow* problem, we construct a numerically, highly efficient *network* method to determine f
- We develop such a scheme by making detailed use of graph theoretic notions
- We start out by introducing some definitions

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DEFINITIONS OF NETWORK TERMS

- Each ***arc*** is directed and so for an arc (i, j),

$$f_{ij} \geq 0$$

- A ***forward*** arc at a node i is one that leaves the node i to some node j and is denoted by (i, j)

- A ***backward*** arc at node i is one that enters node i from some node j and is denoted by (j, i)

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DEFINITIONS OF NETWORK TERMS

- A *path* connecting node i to node j is a *sequence* of arcs that starts at node i and terminates at node j
  - we denote a path by

$$\mathscr{P} = \{ (i, k), (k, l), \ldots, (m, j) \}$$

  - in the example network
    - $\{ (1,2),(2,5),(5,4) \}$ is a path from 1 to 4
    - $\{ (1,3), (3,4) \}$ is another path from 1 to 4

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DEFINITIONS OF NETWORK TERMS

- A *cycle* is a path with the condition $i = j$, i.e.,

$$\mathcal{P} = \{ (i, k), (k, l), \ldots, (m, i) \}$$

- We denote the set of nodes of the network by $\mathcal{N}$
  - the definition is

$$\mathcal{N} = \{ i : i \text{ is a node of the network} \}$$

  - In the example network

$$\mathcal{N} = \{ s, 1, 2, 3, 4, 5, t \}$$

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NETWORK CUT CAPACITY

- A *cut* is a partitioning of nodes into two distinct subsets $\mathcal{S}$ and $\mathcal{T}$ with the properties

$$\mathcal{N} = \mathcal{S} \cup \mathcal{T} \text{ and } \mathcal{S} \cap \mathcal{T} = \varnothing$$

- We are interested in cuts with the property that

$$s \in \mathcal{S} \text{ and } t \in \mathcal{T}$$

- We say that the sets $\mathcal{S}$ and $\mathcal{T}$ provide an $s-t$ cut; in the example network,

$$\mathcal{S} = \{ s, 1, 2 \} \quad \text{and} \quad \mathcal{T} = \{ 3, 4, 5, t \}$$

provide an $s-t$ cut

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NETWORK CUT

- The capacity of a cut is

$$K(\mathcal{S},\mathcal{T}) = \sum_{\substack{s \in \mathcal{S} \\ t \in \mathcal{T}}} k_{st}$$

- In the example network with

$$\mathcal{S} = \{ s, 1, 2 \} \quad \text{and} \quad \mathcal{T} = \{ 3, 4, 5, t \}$$

we have

$$K(\mathcal{S},\mathcal{T}) = k_{13} + k_{25}$$

but for the cut with

$$\mathcal{S} = \{ s, 1, 2, 3, 4 \} \quad \text{and} \quad \mathcal{T} = \{ 5, t \}$$

$$K(\mathcal{S},\mathcal{T}) = k_{4,t} + k_{4,5} + k_{3,5} + k_{2,5}$$

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NETWORK CUT

- **Note: arc (5, 4) is directed from a node in $\mathcal{T}$ to a node in $\mathcal{S}$ and is not included in the summation**
- **A *salient characteristic* of the $s - t$ cuts of interest is that when all the arcs in the cut are removed, then *no* path exists from s to t ; consequently, no flow is possible since any flow from s to t must go through the arcs in a cut**
- **The flow is *limited* by the capacity of the cut**

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NETWORK CUT LEMMA

- **For any directed network, the flow f from s to t is constrained by an $s-t$ cut so that**

$$f \leq K(\mathcal{S},\mathcal{T}) \ \ for\ every\ s-t\ cut\ \ set\ \mathcal{S},\mathcal{T}$$

- **Corollaries of this lemma are**

(*i*) $$max\ \ flow \leq K(\mathcal{S},\mathcal{T})\ \ \forall\ \ \mathcal{S},\mathcal{T}$$

and

(*ii*) $$max\ \ flow \leq \min_{\mathcal{S},\mathcal{T}} K(\mathcal{S},\mathcal{T})$$

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MAX – FLOW – MIN – CUT THEOREM

- For any network, the value of the maximal flow from s to t is equal to the minimal cut, i.e., the cut $\mathcal{S}, \mathcal{T}$ with the smallest capacity

- The *max-flow min-cut* theorem allows us, in principle, to find the maximal flow in a network, we find the capacity of each of the cuts and determine the cut with the smallest capacity

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MAX FLOW

- The *maximal flow* algorithm is based on the identification of a *path* through which a positive flow from *s* to *t* can be sent – the so-called *flow augmenting path*
- The procedure is continued until no such *flow augmenting path* can be found and therefore we have the *maximal flow*
- The maximal flow algorithm is based on the repeated application of the *labeling procedure*

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LABELING PROCEDURE

- The ***labeling procedure*** **is the basic scheme to determine the maximum flow in a network**
- **The *labeling procedure* is used to find a *flow augmenting path* from s to t**
- **We say that a node j can be *labeled* if and only if flow can be sent from s to t and node j is on a path to make such flow possible**

LABELING PROCEDURE

Step 0 : start with node s

Step 1 : given that node i is already labeled, label node j only if

(i) either there exists an arc (i , j) and

$$f_{ij} < k_{ij}$$

(ii) or, there exists an arc (j , i) and

$$f_{ji} > 0$$

Step 2 : if $j = t$, stop; else, return to Step 1

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THE MAX FLOW ALGORITHM

Step *0* : start with a feasible flow

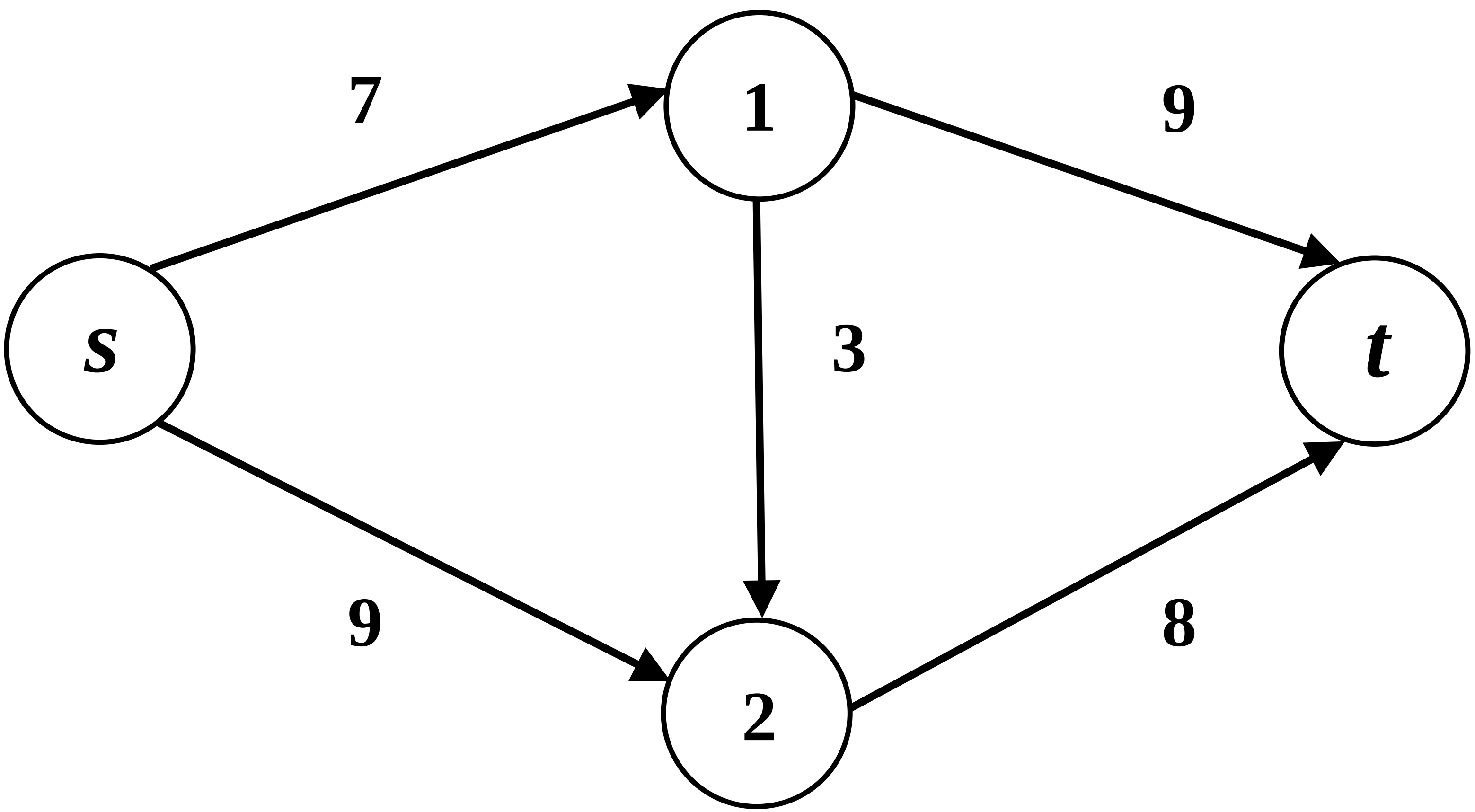
Step 1 : use the *labeling procedure* to find a flow augmenting path

Step 2 : determine the maximum value δ for the largest increase (decrease) of flow on all forward (backward) arcs

Step 3 : use the *labeling procedure* to find a flow augmenting path: if no such path exists, stop; else, go to Step 2

ILLUSTRATIVE EXAMPLE

- Consider the simple network with the flow capacities on each arc indicated



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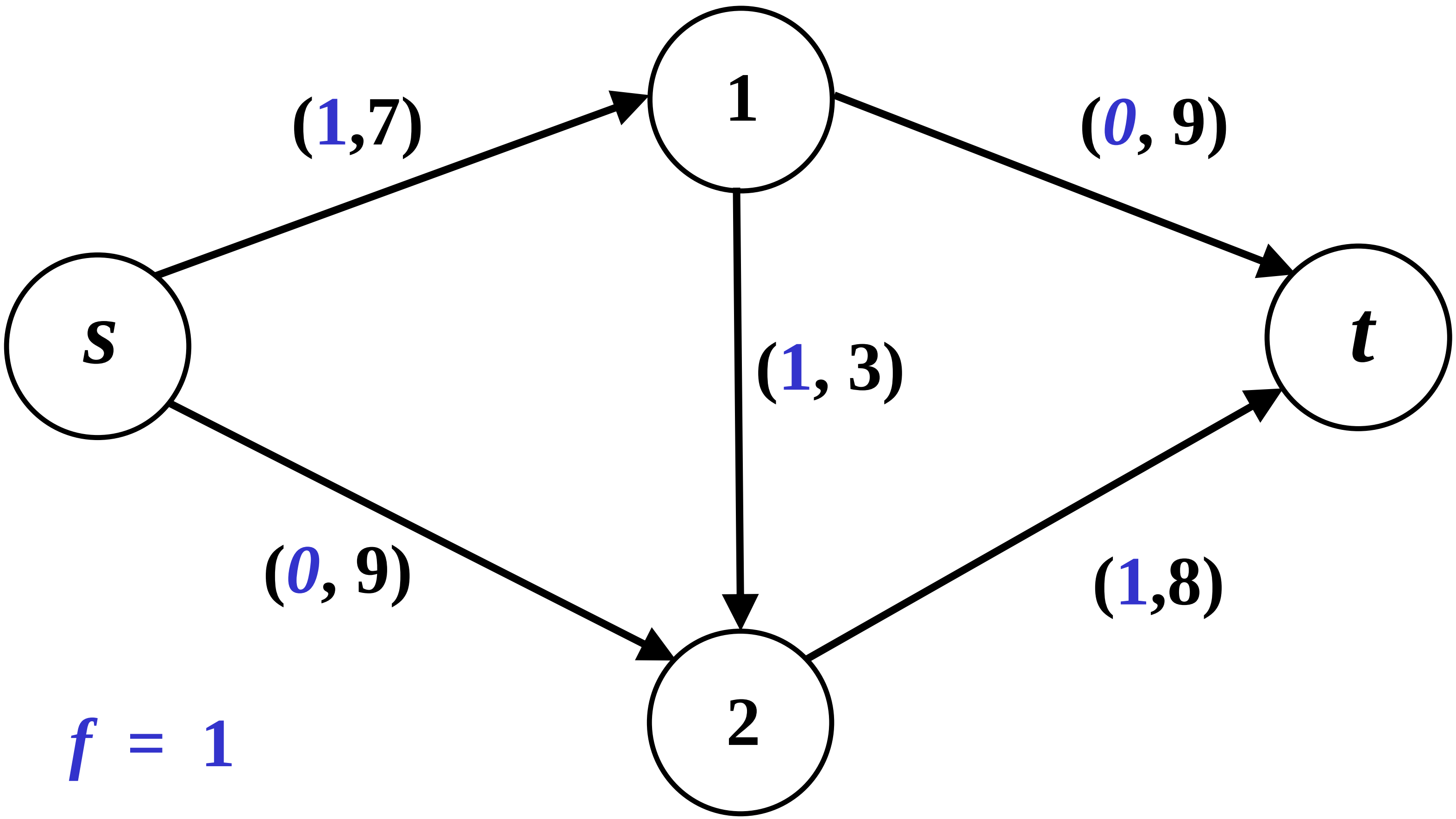
ILLUSTRATIVE EXAMPLE

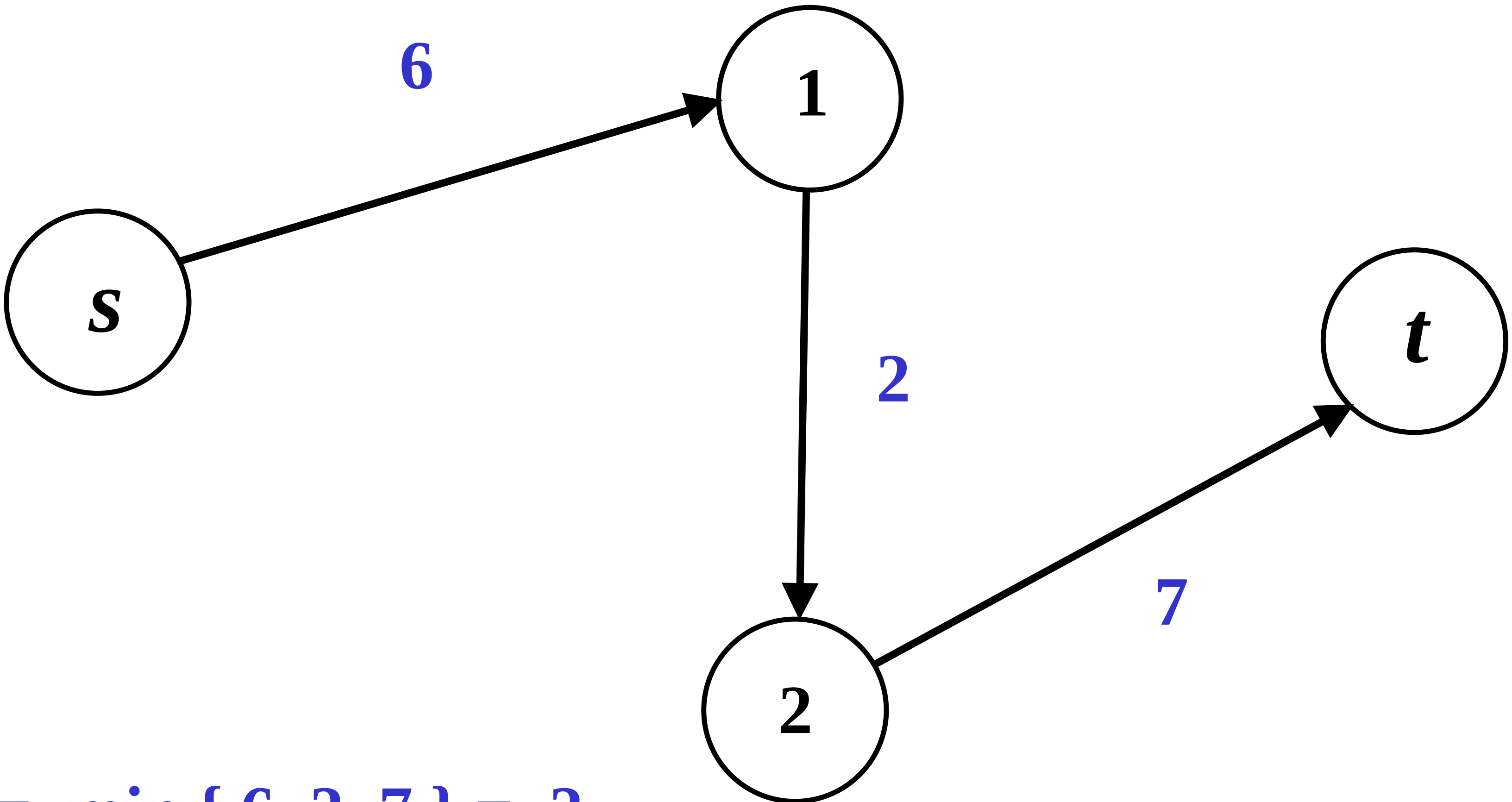
- We initialize the network with a flow 1



f = 1

ILLUSTRATIVE EXAMPLE

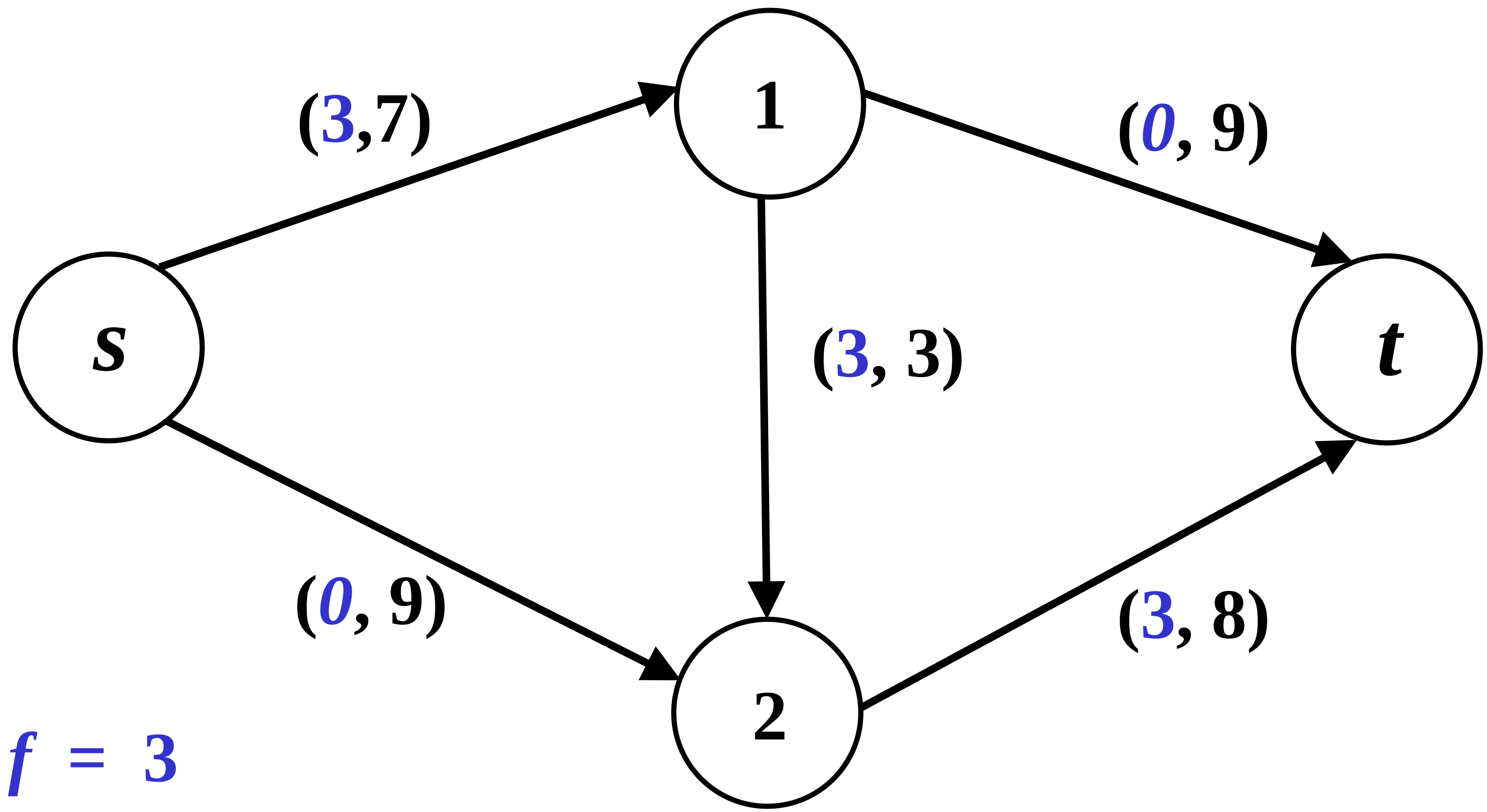
- We apply the labeling procedure



$f = \min\{6, 2, 7\} = 2$

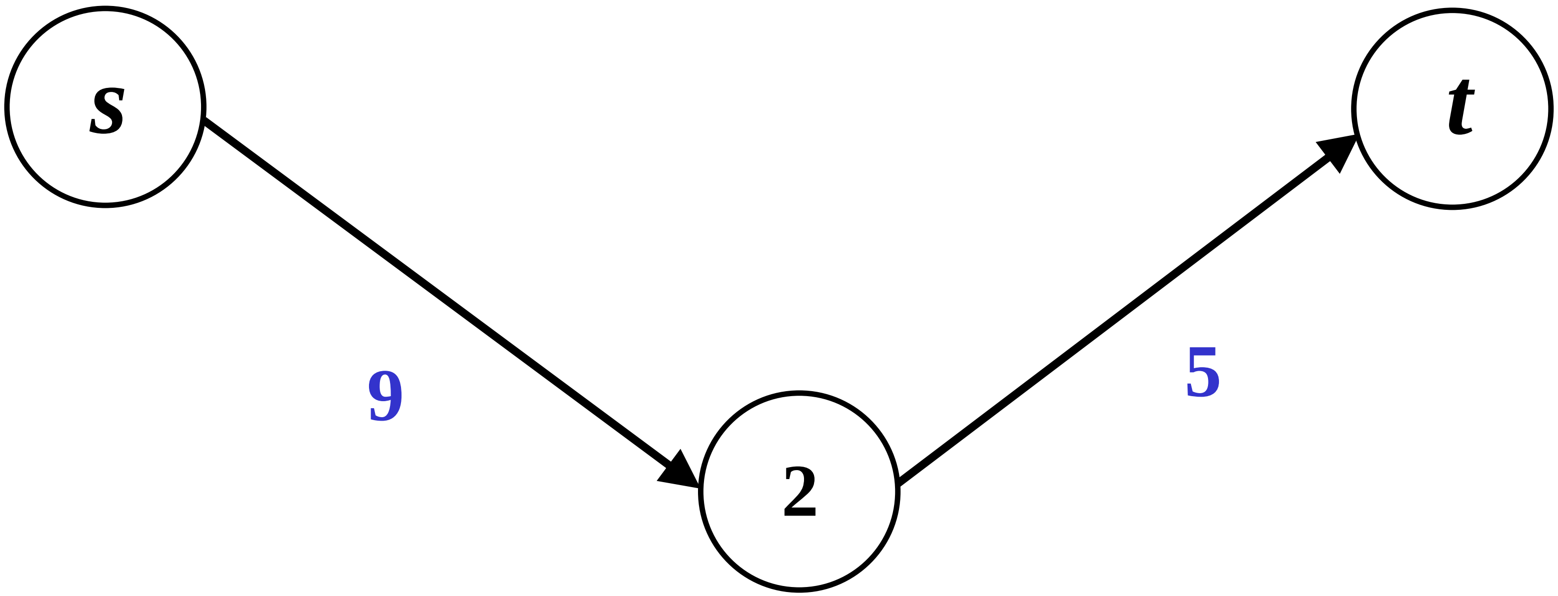
ILLUSTRATIVE EXAMPLE

- **Consider the simple network with the flow and the capacity on each arc (i, j) indicated by (f_{ij}, k_{ij})**



ILLUSTRATIVE EXAMPLE

- We repeat application of the labeling procedure



$f = min \{ 5, 9 \} = 5$

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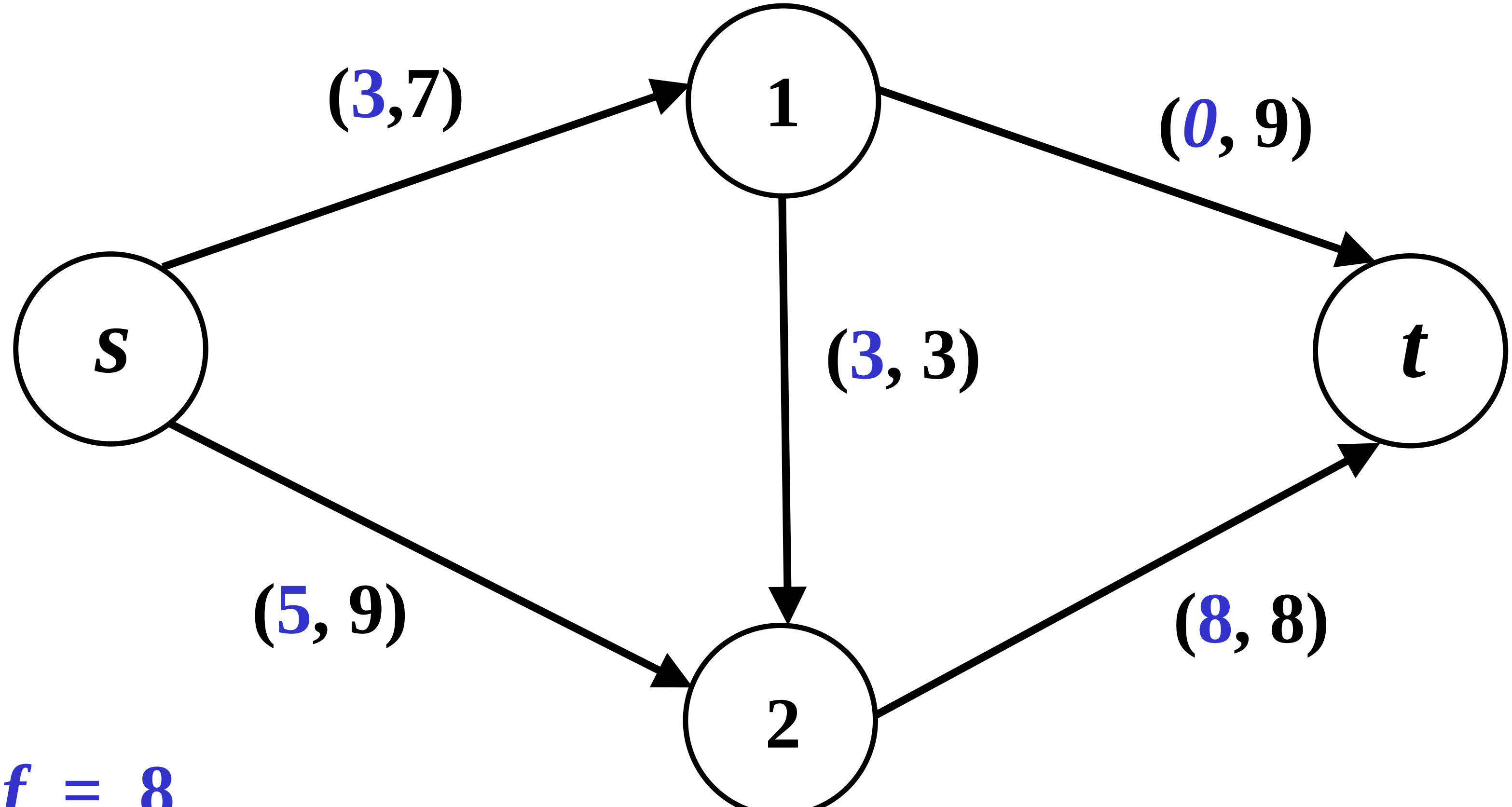
ILLUSTRATIVE EXAMPLE

- We increase the flow by 5



$f = 8$

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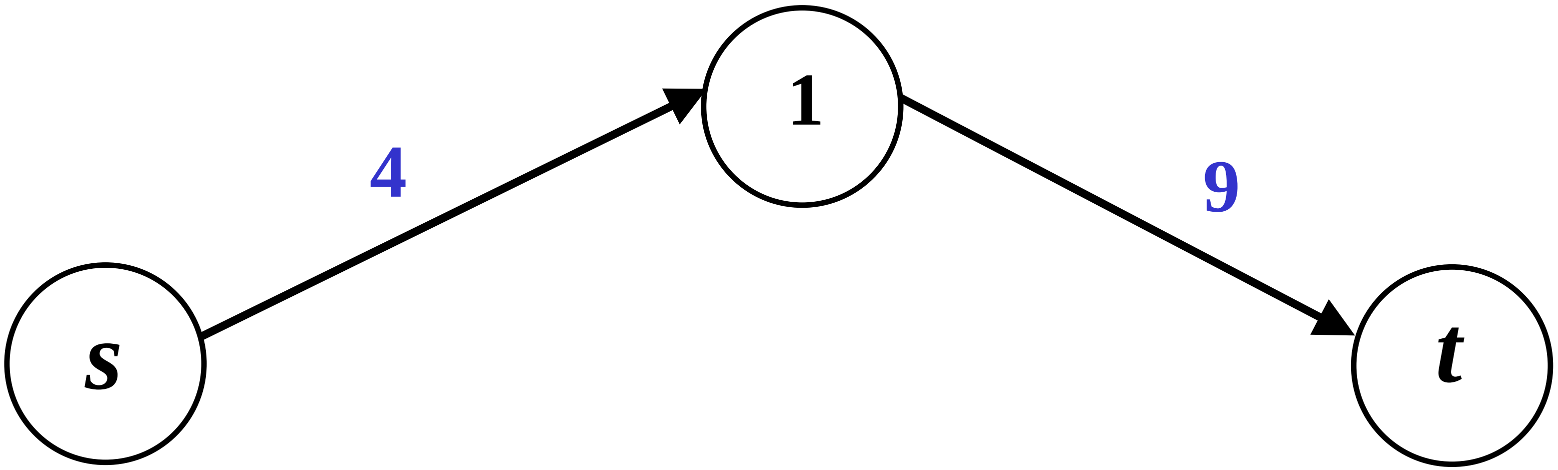
ILLUSTRATIVE EXAMPLE

- We repeat application of the labeling procedure



$$f = min\{4, 9\} = 4$$

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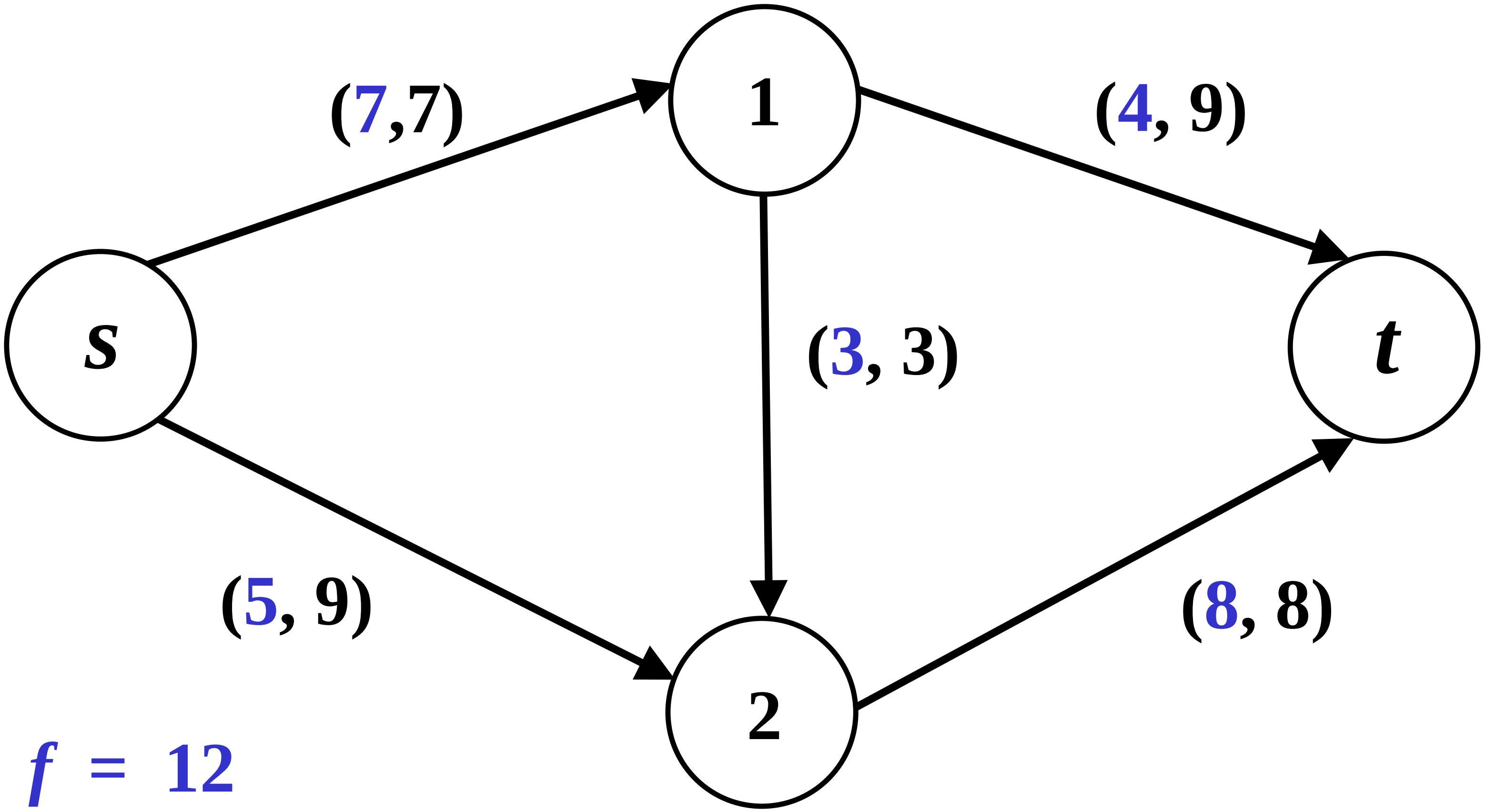
ILLUSTRATIVE EXAMPLE

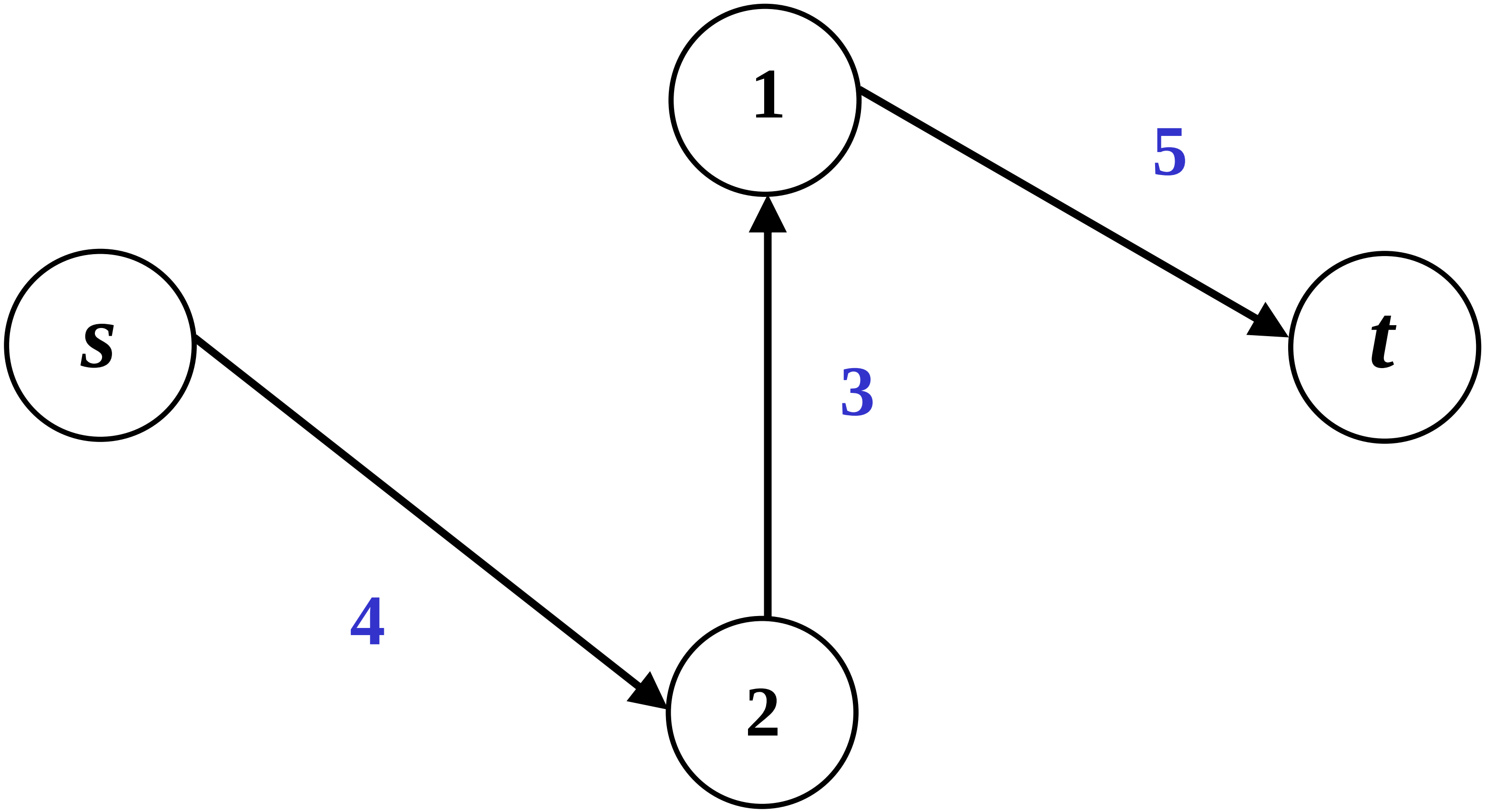
- We increase the flow by 4 to obtain



$f = 12$

ILLUSTRATIVE EXAMPLE

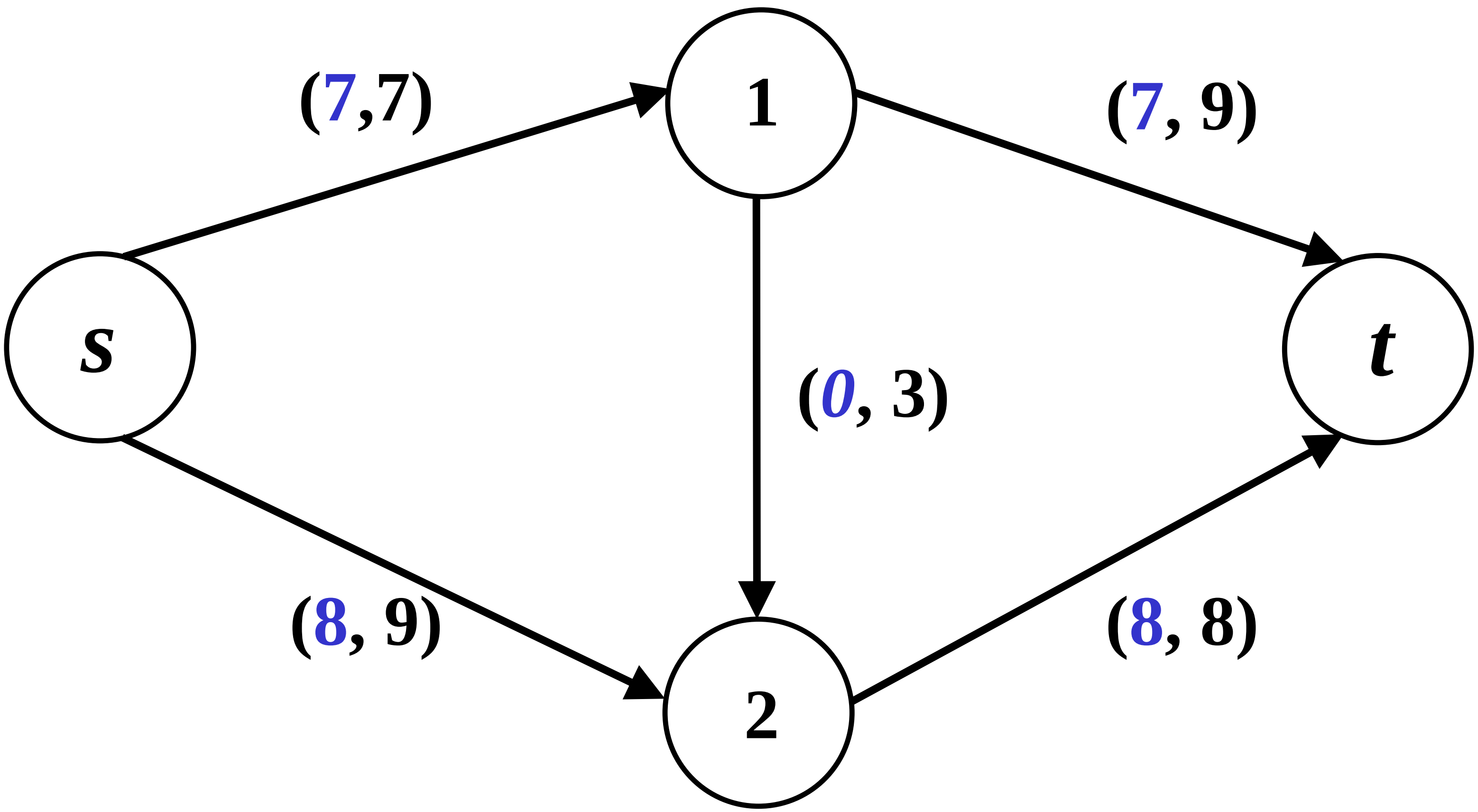
- We repeat application of the *labeling procedure*



$f = min\ \{ 4, 3, 5 \} = 3$

ILLUSTRATIVE EXAMPLE

- We increase the flow by 3



f = 15 with no *flow augmenting path*

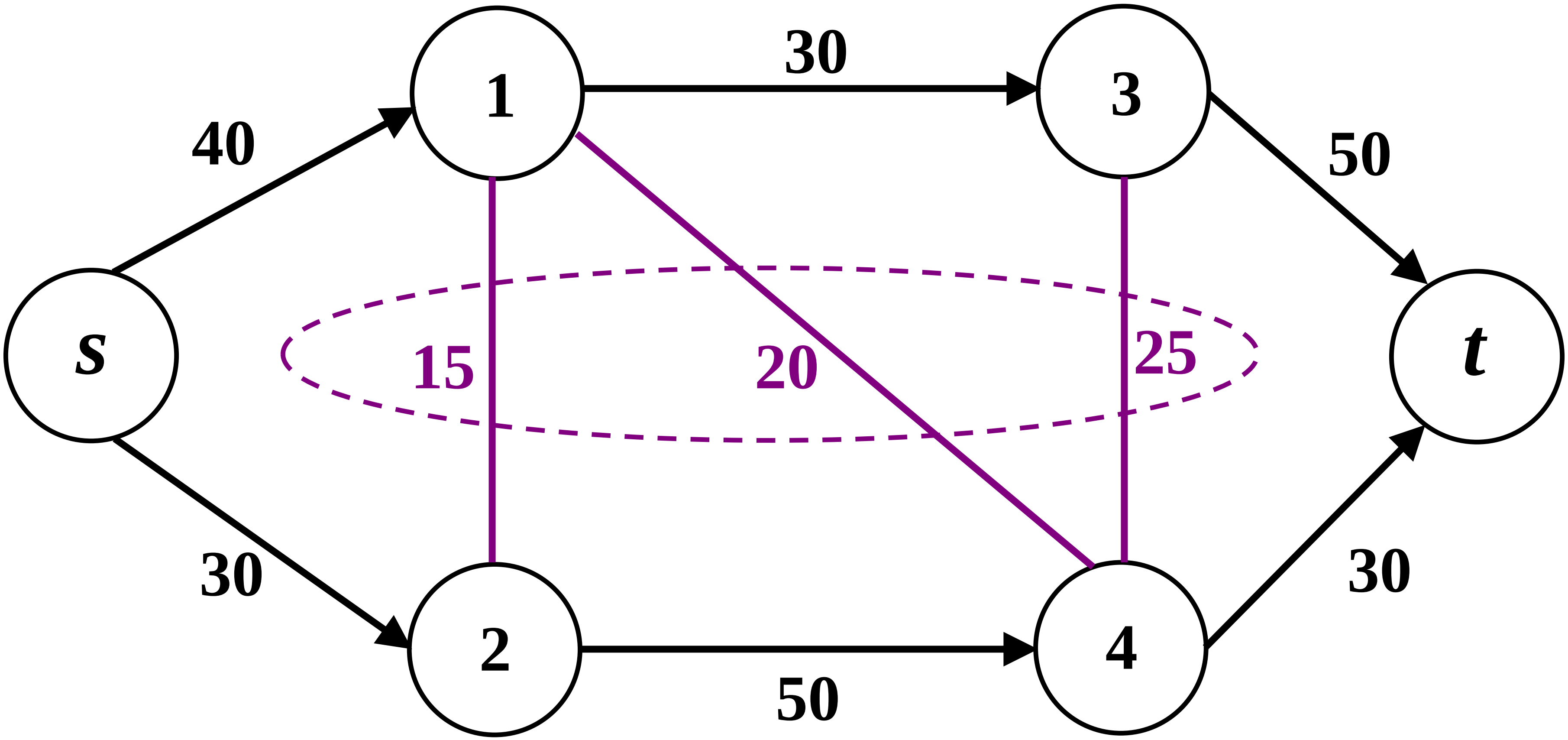
UNDIRECTED NETWORKS

- A network with undirected arcs is called an *undirected network*: the flows on each arc (i, j) with the limit k_{ij} cannot violate the capacity constraints in either direction
- Mathematically, we require

$$\left.\begin{aligned} f_{ij} &\leq k_{ij} \\ f_{ji} &\leq k_{ji} \\ f_{ij} f_{ji} &= 0 \end{aligned}\right\}$$

interpretation of unidirectional flow below capacity limit

EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS



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EXAMPLE OF A NETWORK WITH UNDIRECTED ARCS

- **To make the problem realistic, we may view the capacities as representing traffic flow limits: the directed arcs correspond to *unidirectional* streets and the problem is to place *one-way signs* on each undirected street (i, j) so as to *maximize* the traffic flow from s to t**
- **The procedure is to replace each *undirected arc* by two *directed* arcs (i, j) and (j, i) to determine the maximal $s - t$ flow**

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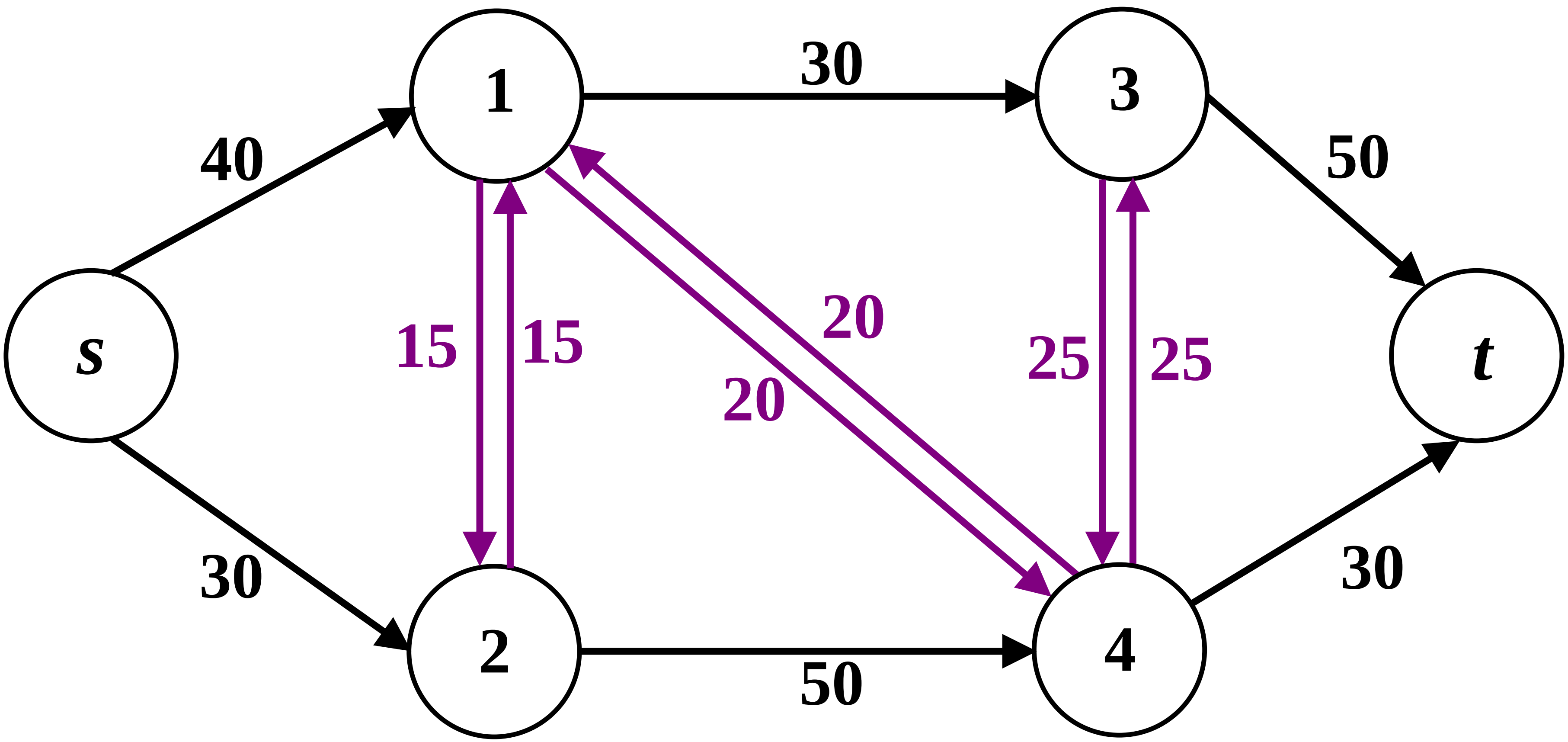
EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS



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EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS

- **We apply the *max flow* scheme to the directed network and give the following interpretations to the flows on the *max flow* bidirectional arcs that are the initially undirected arcs (i, j) : if**

$$f_{ij} > 0 \ , \ f_{ji} > 0 \ \text{and} \ f_{ij} > f_{ji} \ ,$$

  set up the flow from i to j with value $f_{ij} - f_{ji}$ and remove the arc (j, i)
- **The determination of the max flow f for this example is easily determined**

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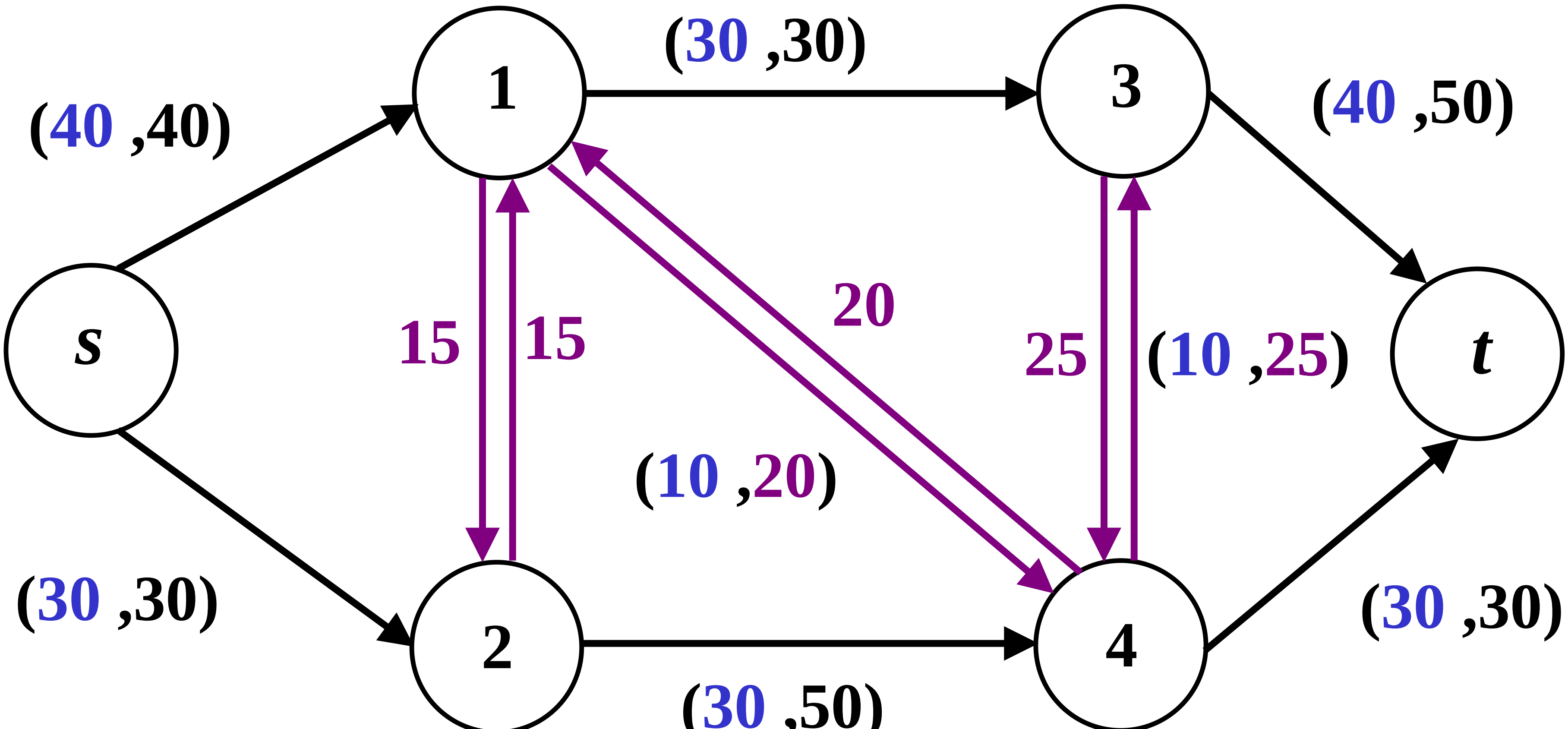
EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS



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EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS : RESULT

flow: $s \rightarrow 1 \rightarrow 3 \rightarrow t$ = 30

flow: $s \rightarrow 2 \rightarrow 4 \rightarrow t$ = 30

flow: $s \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow t$ = 10

and so the maximum flow is 30 + 30 + 10 = 70

one way signs must be put from $1 \rightarrow 4$ **and** $4 \rightarrow 3$ **;**

an alternative path of a flow of 10 is the path:

$s \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow t$ **, which requires one-way**

signs from $1 \rightarrow 2$ **and** $4 \rightarrow 3$

NETWORKS WITH MULTIPLE SOURCES AND MULTIPLE SINKS

- We next consider a network with several supply and several demand points
- We introduce a super *source* $\hat{s}$ linking to all the *sources* and a super *sink* $\hat{t}$ linking all the *sinks*
- We can consequently apply the *max flow* algorithm to the modified network

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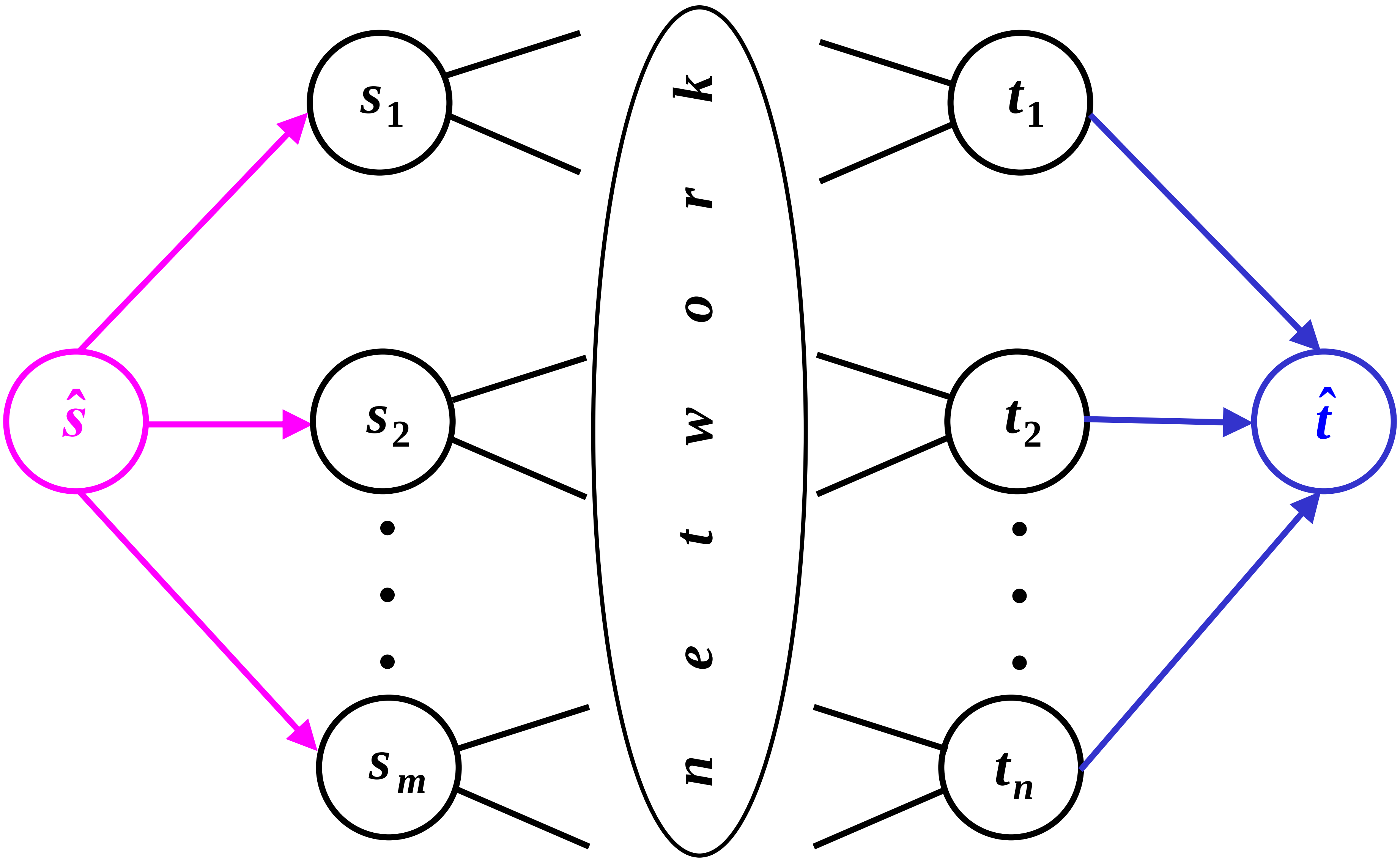
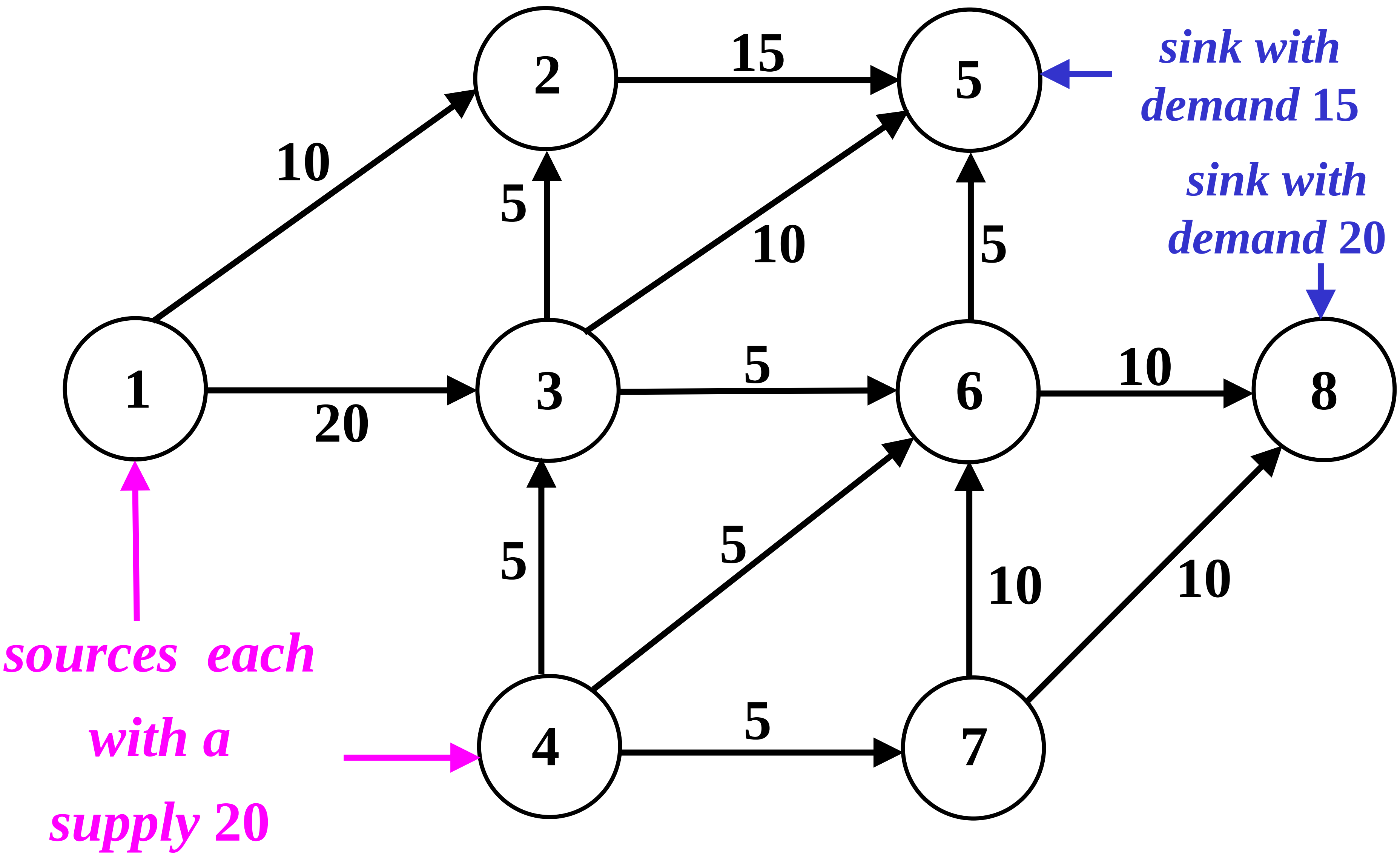
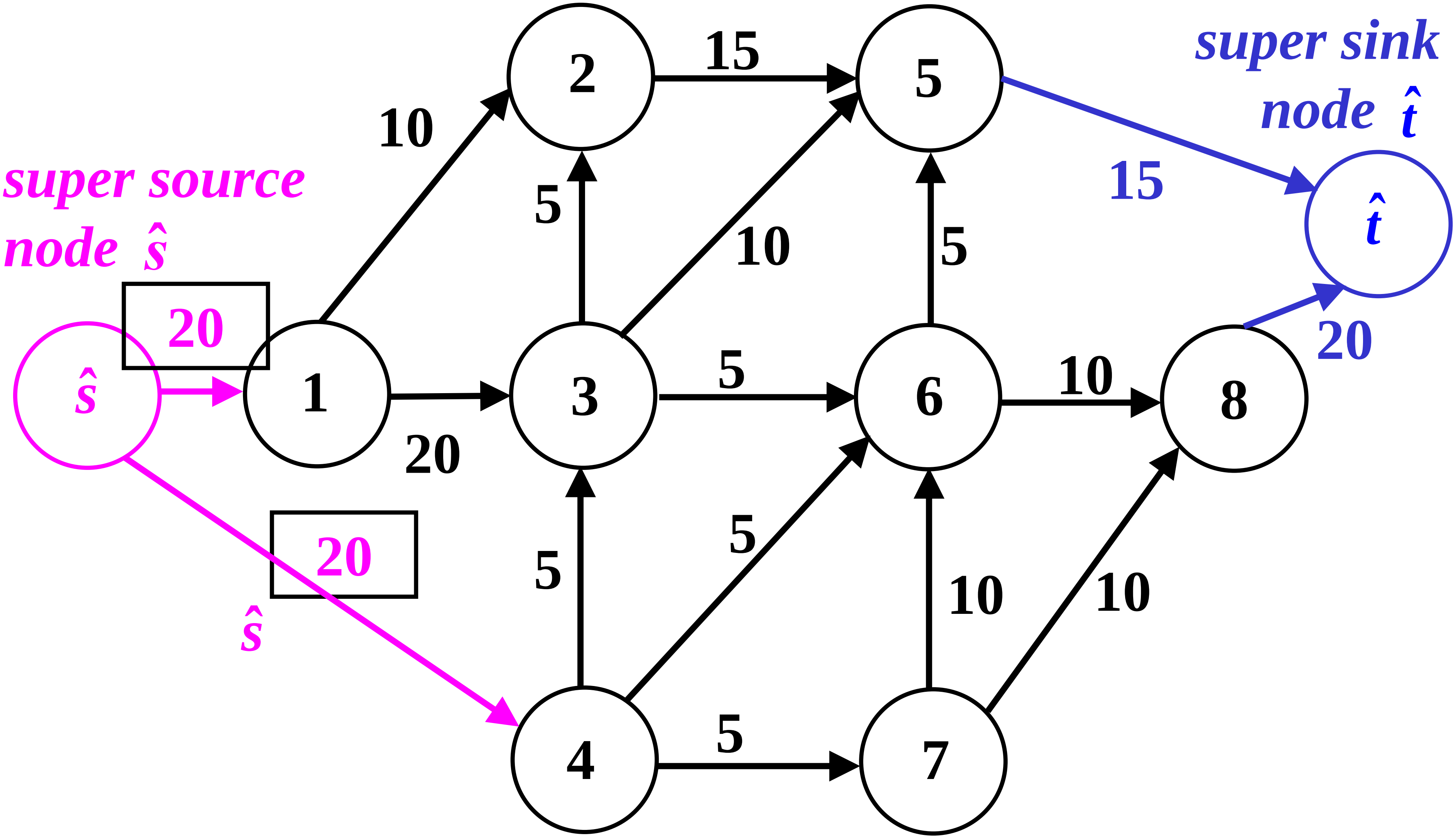
NETWORKS WITH MULTIPLE SOURCES AND MULTIPLE SINKS



MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE



MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE



MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

- The transshipment problem is feasible if and only if the maximal $\hat{s}-\hat{t}$ flow f satisfies

$$f = \sum_{\text{sinks}} demands$$

- We need to show that
  - the transshipment problem is infeasible since the network cannot accommodate the total demand of 35
  - the smallest shortage for this problem is 5

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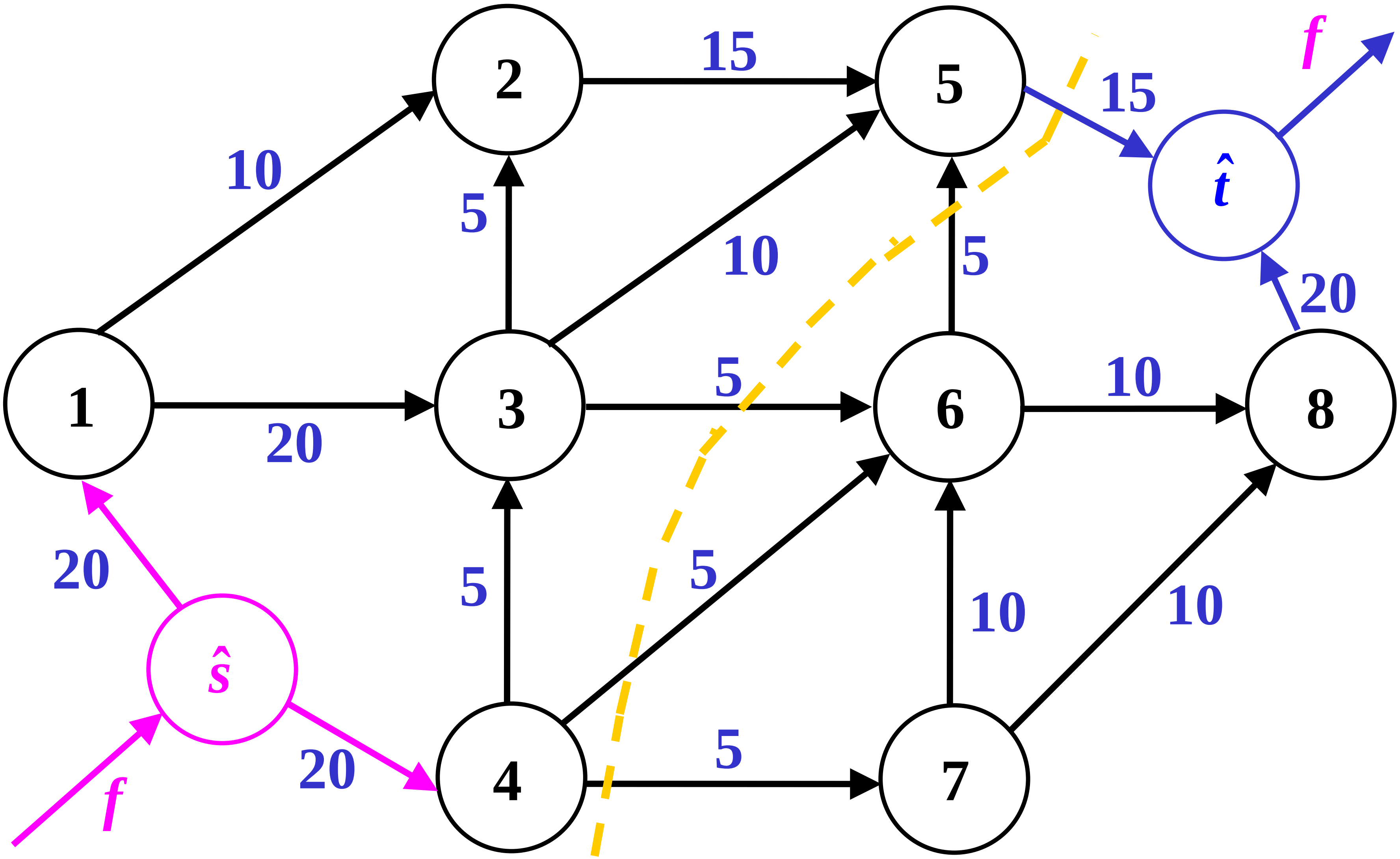
MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE



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MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

- **The minimum cut is shown and has capacity**

  $$15 + 5 + 5 + 5 = 30 ;$$

  the maximum flow is, therefore, 30

- **Since the maximum flow fails to meet the total demand of 35 units by the super sink, the problem is infeasible; the minimum shortage is 5**

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APPLICATION TO MANPOWER SCHEDULING

- **Consider the case of a company that must complete its 4 engineering projects within 6 months**

project	*earliest start month*	*latest finish month*	*manpower requirements (man month)*
A	**1**	**4**	**6**
B	**1**	**6**	**8**
C	**2**	**5**	**3**
D	**1**	**6**	**4**

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APPLICATION TO MANPOWER SCHEDULING

- **There are the following additional constraints:**
  - **the company has only 4 engineers**
  - **at most 2 engineers may be assigned to any one project in a given month**
  - **no engineer may be assigned to more than one project at any time**
- **The question is whether there is a *feasible assignment* and, if so, determine the *optimal assignment***

APPLICATION TO MANPOWER SCHEDULING

- **The solution approach is to set up the problem as a transshipment network**
  - **the *sources* are the 6 months of engineer labor**
  - **the *sinks* are the 4 projects that must be done**
  - **an arc (i, j) is introduced whenever a feasible assignment of the engineers who work in month i can be made to project j with**

$$k_{ij} = 2 \qquad i = 1, 2, \ldots, 6 \;, \quad j = A, B, C, D$$

  - **there is no arc $(1, C)$ since project C cannot start before month 2**

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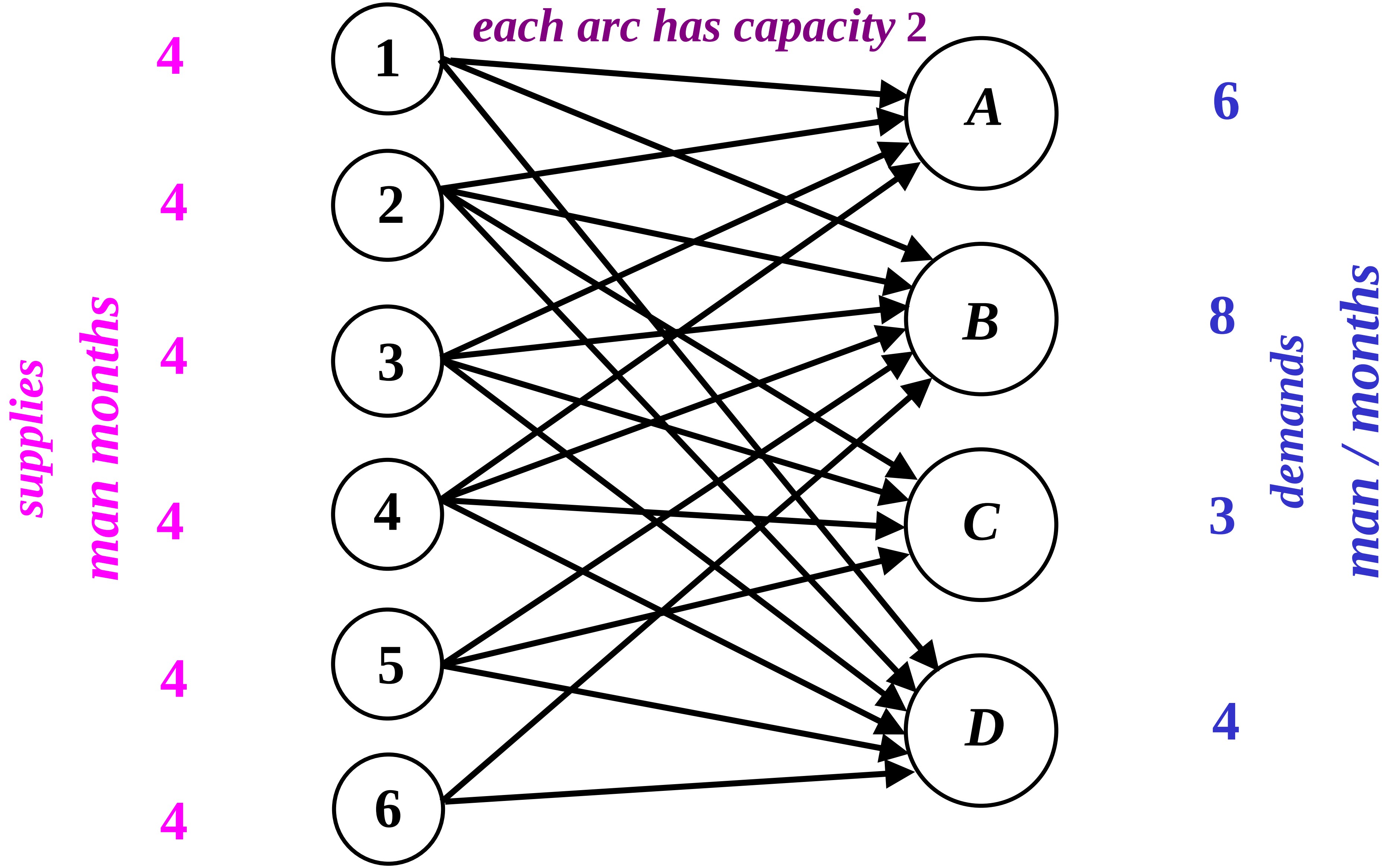
APPLICATION TO MANPOWER SCHEDULING



APPLICATION TO MANPOWER SCHEDULING

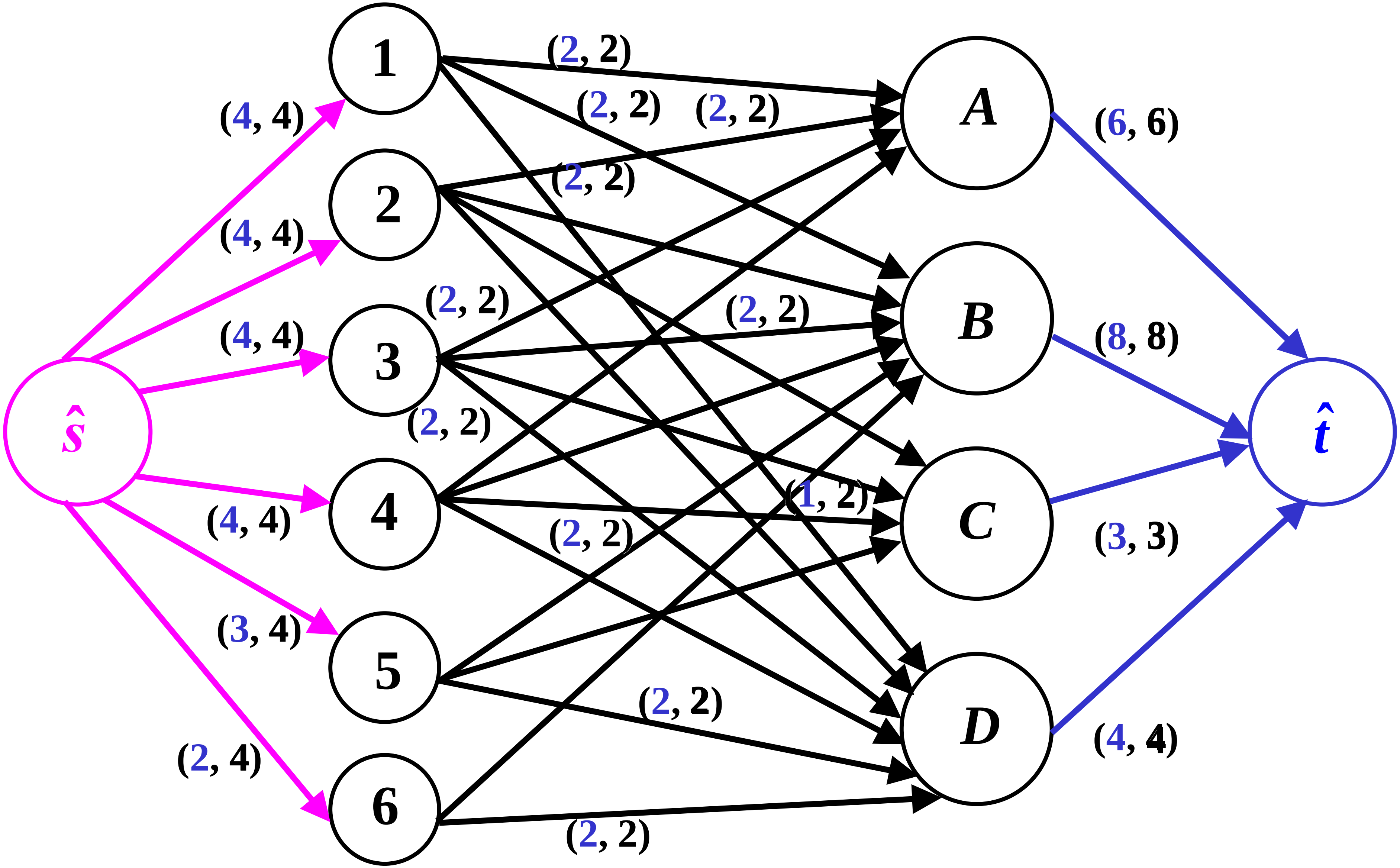
- **The transshipment problem is feasible if the total demand**

$$6 + 8 + 3 + 4 = 21$$

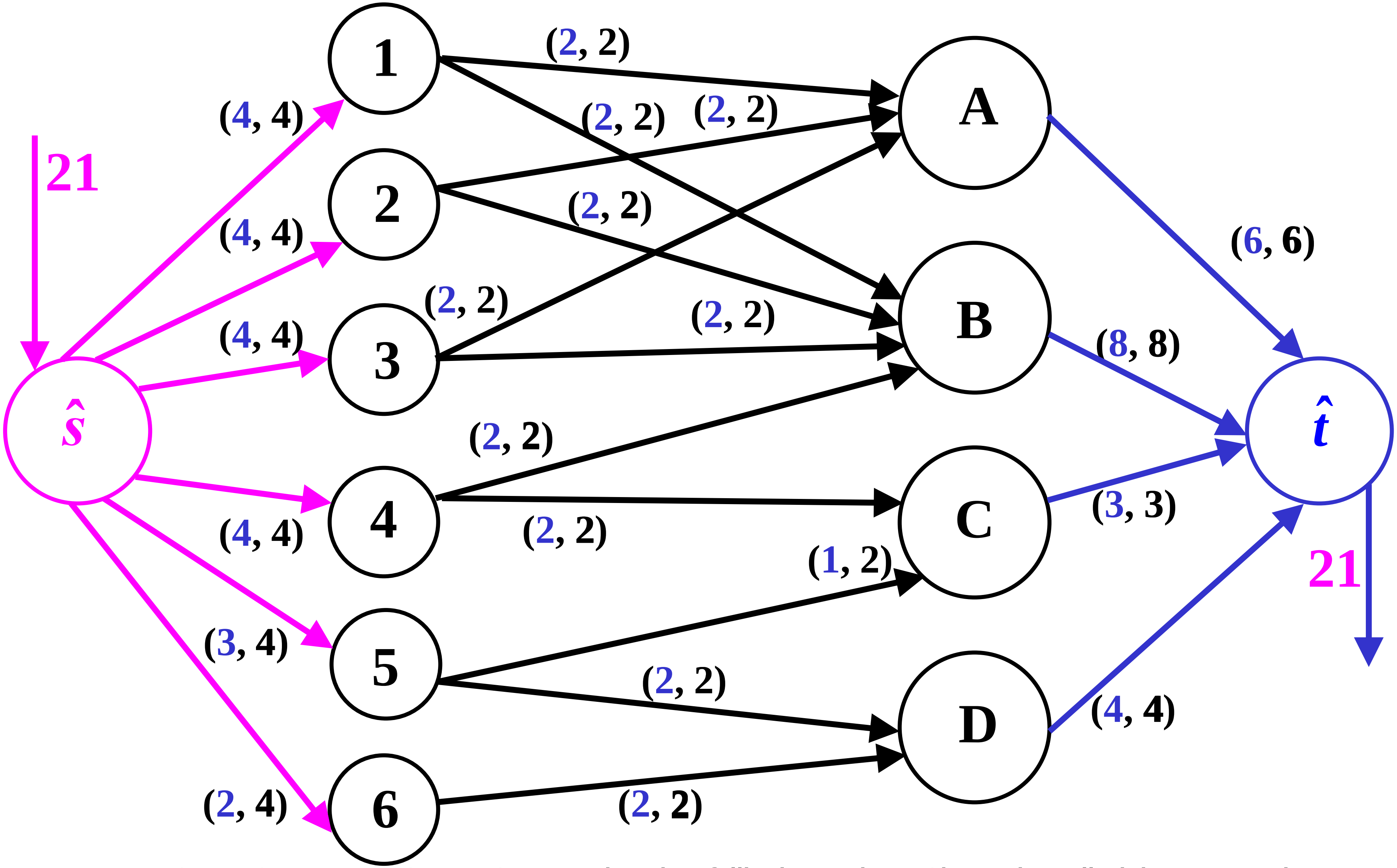
  can be met

- **We determine whether a feasible schedule exists and if so, we find it**

APPLICATION TO MANPOWER SCHEDULING



APPLICATION TO MANPOWER SCHEDULING



SHORTEST ROUTE PROBLEM

- **The problem is to determine the *shortest path* from $s = 1$ to $t = n$ in a network with the set of nodes**

$$\mathcal{N} = \{1, 2, \ldots, n\}$$

and the set of arcs $\{(i, j)\}$, where for each arc (i, j)

$$d_{ij} = \textit{distance or transit time}$$

- **The determination of the shortest path from 1 to n requires the specification of the path**

$$\{ (1, i_1), (i_1, i_2), \ldots, (i_q, n) \}$$

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SHORTEST ROUTE PROBLEM

- We can write an *LP* formulation of this problem in the form of a ***transshipment problem***:

  ship 1 unit from node 1 to node n by minimizing the shipping costs using the costs

$$d_{ij} = \begin{cases} \textit{shipping costs for } 1 \textit{ unit from } i \textit{ to } j \\ \text{¥} \textit{ whenever } i \textit{ and } j \textit{ are not directly connected} \end{cases}$$

- But, in practice, we use the *Dijkstra scheme solution*

THE DIJKSTRA ALGORITHM

- The solution is very efficiently performed using ***the Dijkstra algorithm***
- The assumptions are
  - d_{ij} is given for each pair of connected nodes
  - $d_{ij} \geq 0$
- The scheme is, basically, a label assignment procedure, which assigns nodes with either a ***permanent*** or a ***temporary*** label

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THE DIJKSTRA ALGORITHM

- The *temporary* label of a node i is an upper bound on the shortest distance from node 1 to node i
- The *permanent* label is the actual shortest distance from node 1 to node i
- A temporary label becomes permanent when we find the tightest upper bound, i.e., the shortest distance

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THE DIJKSTRA ALGORITHM

Step *0*: assign the *permanent* label *0* to node 1

Step 1: assign *temporary* labels to all the other nodes

- d_{1j} **if node j is directly connected to node 1**
- ∞ **if node j is not directly connected to node 1**

and select the minimum of the *temporary* labels and declare it *permanent*; in case of ties, the choice is arbitrary (but requires a rule)

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THE DIJKSTRA ALGORITHM

Step 2 : let ℓ be the node most recently assigned a *permanent* label and consider each node j with a *temporary* label; recompute each label

$$\min \left\{ \begin{array}{c} \textit{temporary label} \\ \textit{of node } j \end{array} , \begin{array}{c} \textit{permanent label} \\ \textit{of node } \ell \end{array} + d_{\ell j} \right\}$$

Step 3 : select the smallest of the *temporary* labels and declare it *permanent* ; in case of ties, the choice is arbitrary (but we need a rule)

Step 4 : if the selected node is n , stop; else, go to Step 2

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THE DIJKSTRA ALGORITHM

- The shortest path is obtained by retracing the sequence of nodes with permanent labels starting at node n and returning back to node 1
- The path is then given in the forward direction starting from node 1 and ending at node n

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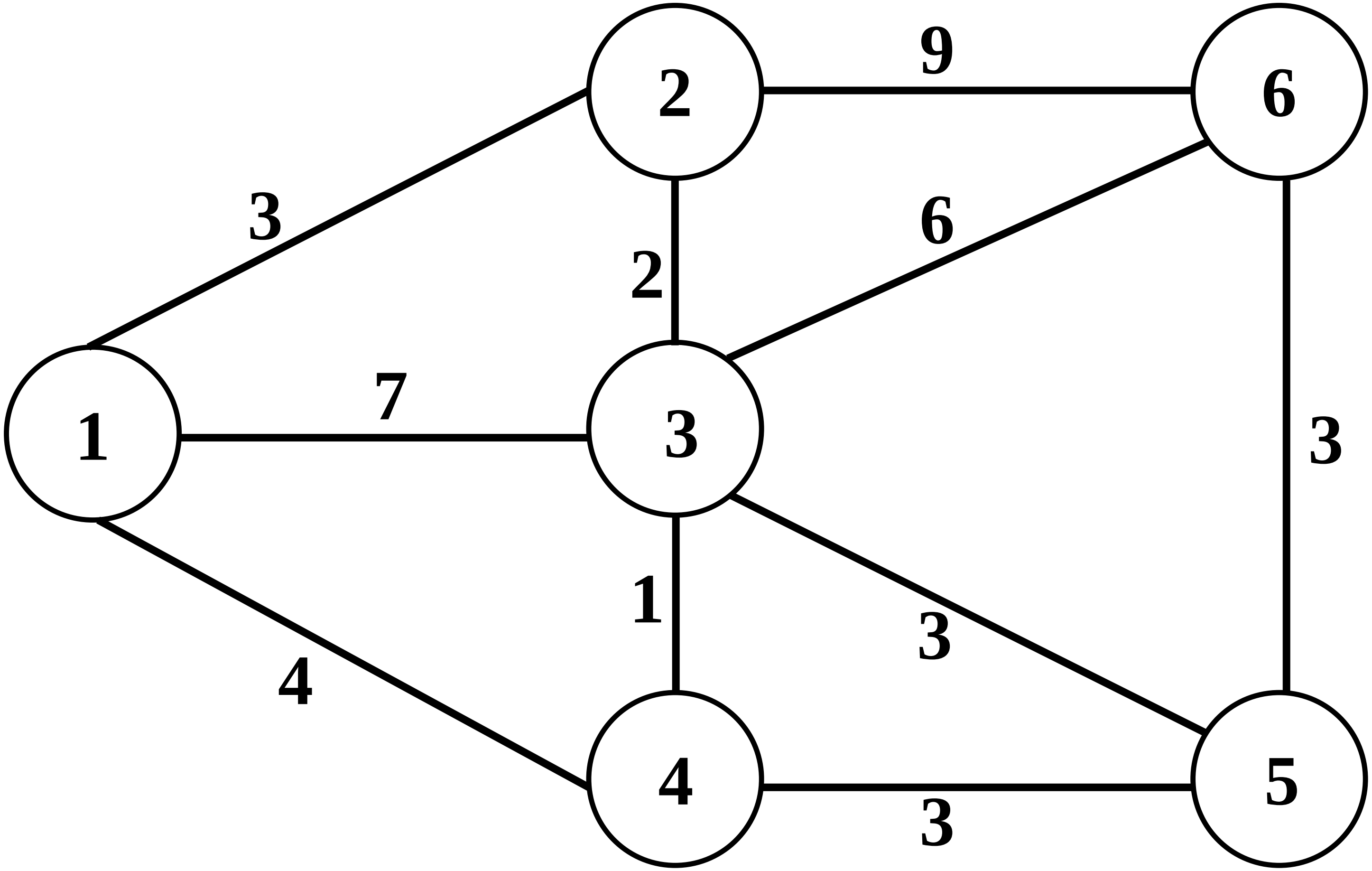
EXAMPLE : SHORTEST PATH

- Consider the undirected network



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EXAMPLE : SHORTEST PATH

- **The problem is to**
  - **find the shortest path from 1 to 6**
  - **compute the length of the shortest path**
- **We apply the Dijkstra algorithm and assign iteratively a *permanent* label to each node**

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EXAMPLE : SHORTEST PATH

$\boxed{0}$

Steps 0 **and** 1 : $\mathscr{L}(0) = [0,3,7,4,\infty,\infty]$

initial label

$\boxed{1}$

Step 2 : $\mathscr{L}(1) = [0,3,5,4,\infty,12]$

label in iteration 1

$\boxed{2}$

Steps $2,3$ **and** 4 : $\mathscr{L}(2) = [0,3,5,4,7,12]$

label in iteration 2

$\boxed{3}$

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EXAMPLE : SHORTEST PATH

Steps 2,3 and 4 : $\mathscr{L}(3) = [0,3,5,4,7,11]$

4

label in iteration 3

Steps 2,3 and 4 : $\mathscr{L}(4) = [0,3,5,4,7,10]$

5

label in iteration 4

$\mathscr{L}(4) = [0,3,5,4,7,10]$

6

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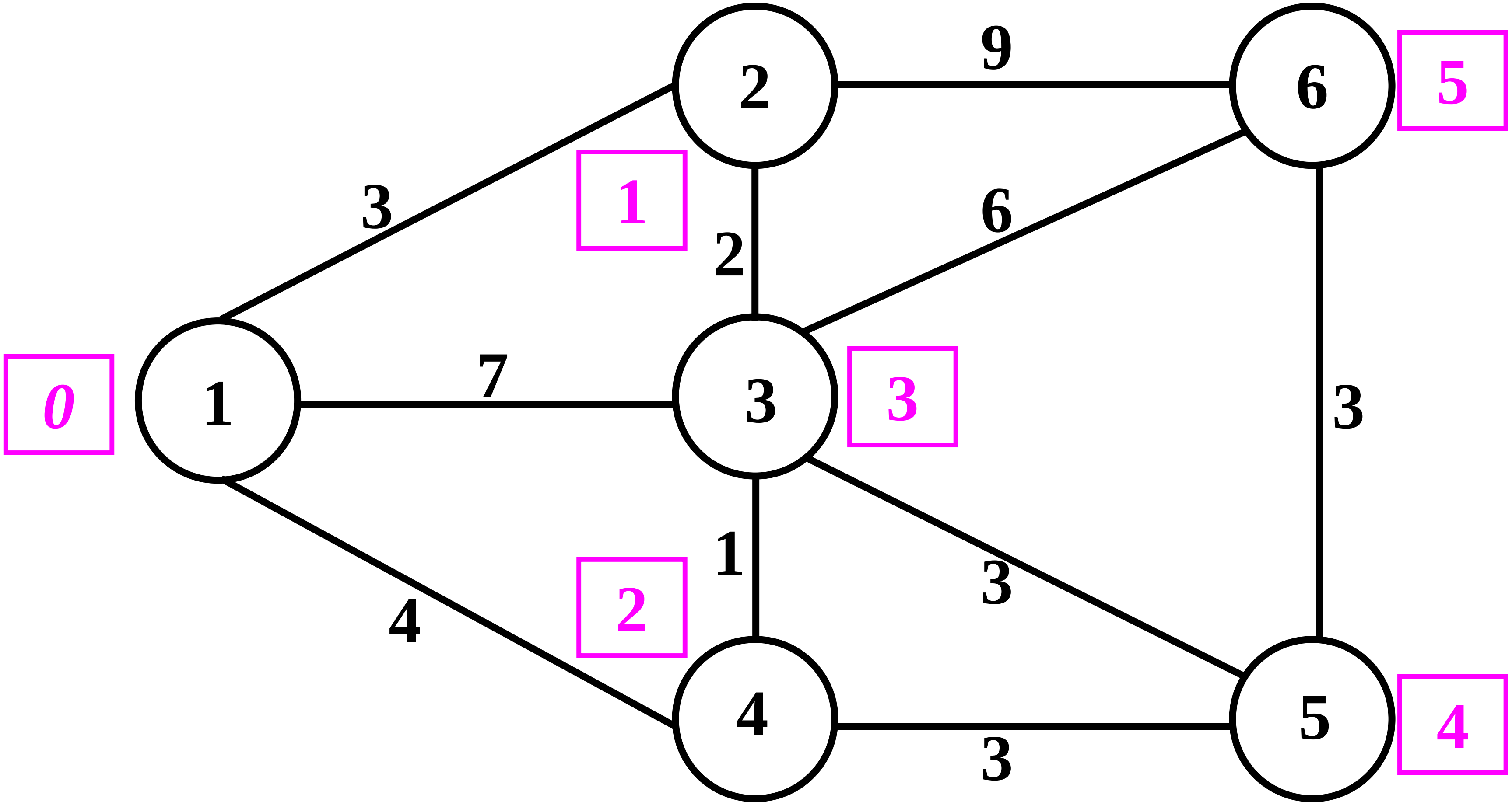
EXAMPLE : SHORTEST PATH



- The shortest distance is 10 obtained with the path

{ (1, 4) , (4, 5) , (5, 6) }

PATH RETRACING

- We retrace the path from n back to 1 using the scheme:

  pick node j preceding node n as the node with the property

$$\textit{permanent label of node } j \quad + \quad d_{jn} \quad = \quad \textit{shortest distance}$$

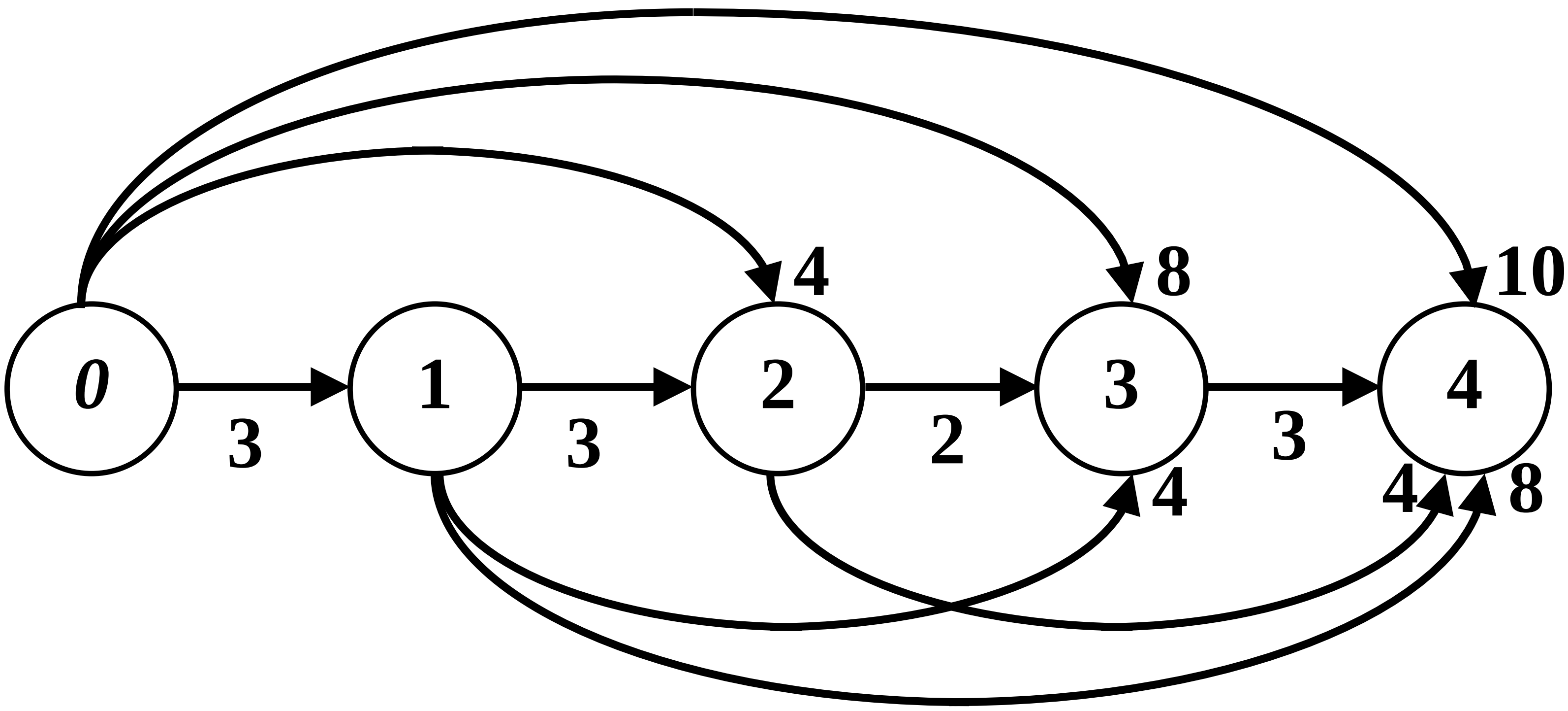
- In the retracing scheme, certain nodes may be skipped

SHORTEST PATH BETWEEN ANY TWO NODES

- The Dijkstra algorithm may be applied to compute the *shortest distance* between *any* pair of nodes i , j by taking i as the *source* node and j as the *sink* node

- We give as an example the following five – node network

EXAMPLE : FIVE – NODE NETWORK



EXAMPLE : FIVE – NODE NETWORK

$$\mathscr{L}(0) = \left[\mathit{0}, 3, 4, 8, 10\right]$$

$$\boxed{\mathit{0}}$$

$$\mathscr{L}(1) = \left[\mathit{0}, 3, 4, 7, 10\right]$$

$$\boxed{1}$$

$$\mathscr{L}(2) = \left[\mathit{0}, 3, 4, 6, 8\right]$$

$$\boxed{2}$$

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EXAMPLE : FIVE – NODE NETWORK

$$\mathcal{L}(3) = \left[\mathit{0}, 3, 4, 6, 8 \right]$$

3

determines the shortest distance from *0* to every other node

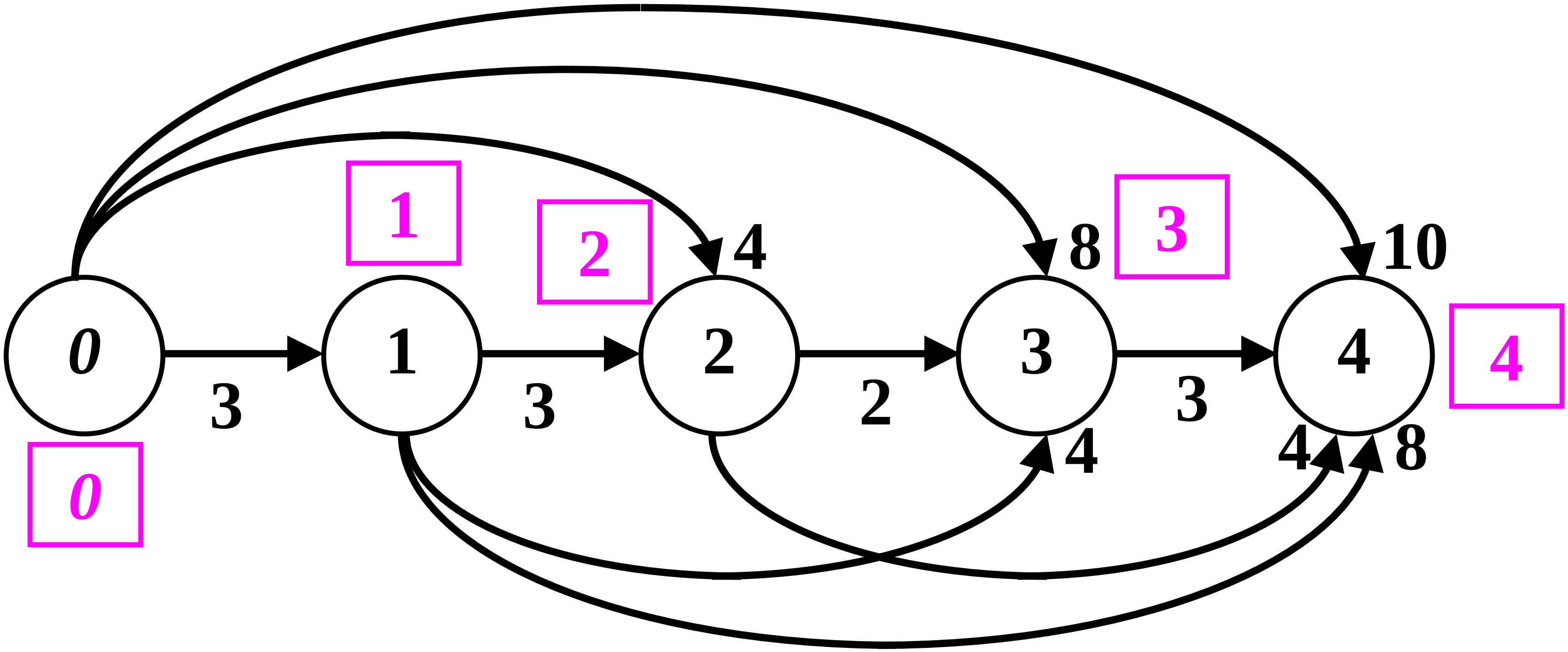
We retrace the path to get

$$8 = \underbrace{4}_{node\,2} + \underbrace{d_{24}}_{4}$$

and so the path is

$$\mathit{0} \rightarrow 2 \rightarrow 4$$

EXAMPLE : FIVE – NODE NETWORK



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APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- We consider the problem of old equipment replacement or its continued maintenance
- As equipment ages, the level of maintenance required increases and typically, this results in increased operating costs
- O&M costs may be reduced by replacing aging equipment; however, replacement requires additional capital investment and so higher fixed costs

APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- **The problem is how often to replace equipment so as to minimize the total costs given by**

total costs	=	***capital costs***	+	***O&M costs***
		↑		↑
		fixed		***variable***

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EXAMPLE: EQUIPMENT REPLACEMENT

- **Equipment replacement is planned during the next 5 years**
- **The cost elements are**

$$p_j \quad = \quad \textit{purchase costs in year } j$$

$$s_j \quad = \quad \textit{salvage value of original equipment after } j \textit{ years of use}$$

$$c_j \quad = \quad \textit{O\&M costs in year } j \textit{ of operation of equipment with the property that}$$

$$\ldots \; c_j < c_{j+1} < c_{j+2} < \; \ldots$$

- **We formulate this problem as a *shortest route* problem on a directed network**

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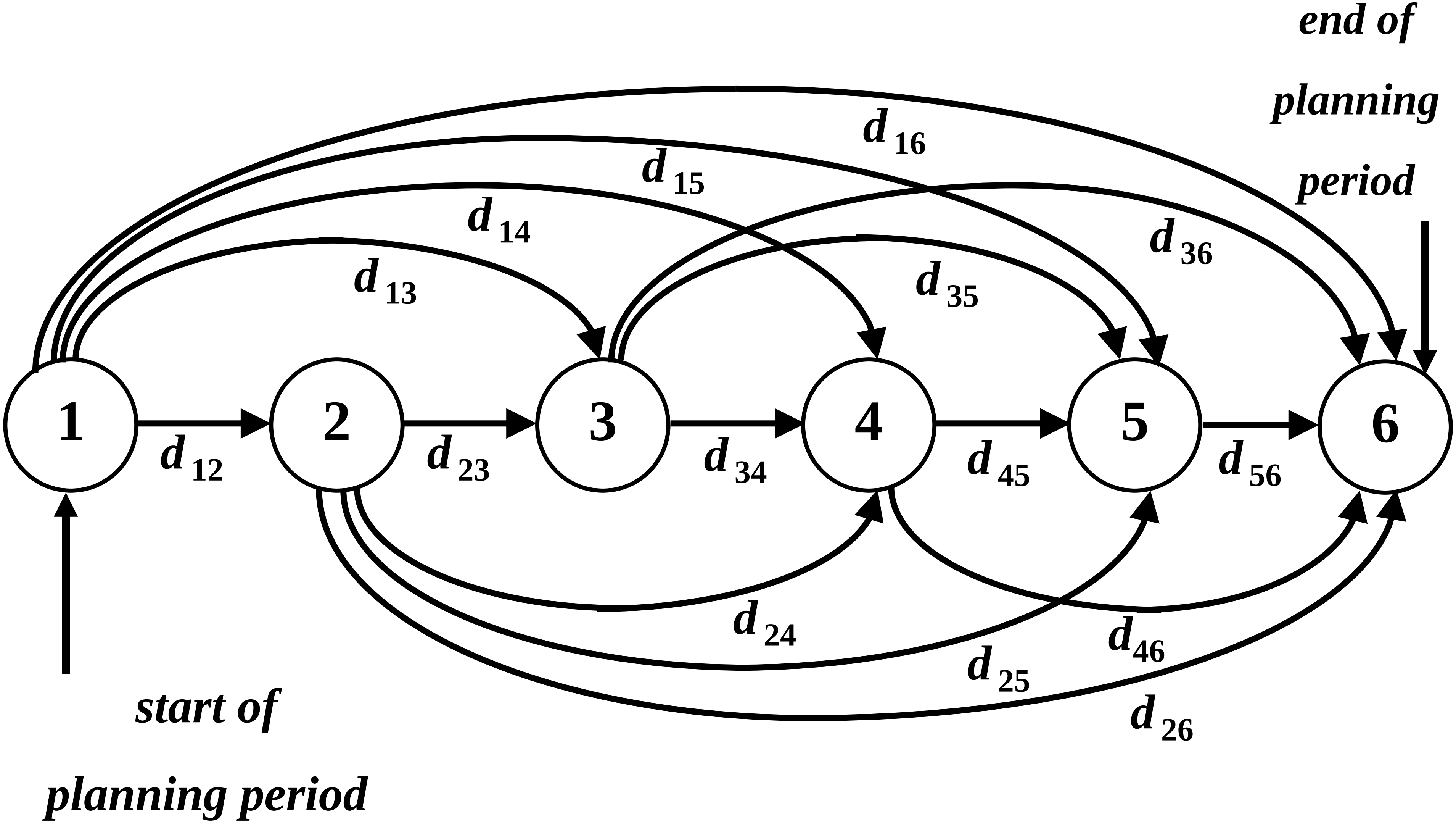
EQUIPMENT REPLACEMENT PROBLEM



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APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

The "distances" d_{ij} are defined to be *finite* if $i < j$, i.e., year i precedes the year j, with

$$d_{ij} = p_i - s_{j-i} + \sum_{\tau=1}^{j-i} c_\tau \qquad j > i$$

p_i	s_{j-i}	$\sum_{\tau=1}^{j-i} c_\tau$
purchase price in year i	*salvage value after j – i years of use*	*O&M costs for j – i years of operation*

APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- For example, if the purchase is made in year 1

$$d_{16} = p_1 - s_5 + \sum_{\tau=1}^{5} c_\tau$$

- The solution is the shortest distance path from year 1 to year 6; if for example the path is

$$\{ (1, 2), (2, 3), (3, 4), (4, 5), (5, 6) \}$$

then the solution is interpreted as the replacement of the equipment each year with

$$total\ costs = \sum_{\tau=1}^{5} p_\tau - 5s_1 + 5c_1$$

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COMPACT BOOK STORAGE IN A LIBRARY

- This problem concerns the storage of books in a limited size library
- Books are stored according to their size, in terms of height and thickness, with books placed in groups of same or higher height; the set of book heights $\{ H_i \}$ is arranged in ascending order with

$$H_1 < H_2 < \ldots < H_n$$

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COMPACT BOOK STORAGE IN A LIBRARY

- Any book of height H_i may be shelved on a shelf of height at least H_i, i.e., $H_i, H_{i+1}, H_{i+2}, \ldots$
- The length L_i of shelving required for height H_i is computed given the thickness of each book; the total shelf area required is $\sum_i H_i L_i$
  - if only 1 height class [corresponding to the tallest book] exists, total shelf area required is the total length of the thickness of all books times the height of the tallest book

COMPACT BOOK STORAGE IN A LIBRARY

- if 2 or more height classes are considered, the total area required is less than the total area required for a single class

- The costs of construction of shelf areas for each height class H_i have the components

  s_i fixed costs [independent of shelf area]

  c_i variable costs / unit area

COMPACT BOOK STORAGE IN A LIBRARY

- For example, if we consider the problem with 2 height classes H_m and H_n with $H_m < H_n$
  - all books of height $\leq H_m$ are shelved in shelf with the height H_m
  - all the other books are shelved on the shelf with height H_n
- The corresponding total costs are

$$\left[s_m + c_m H_m \sum_{j=1}^{m} L_j \right] + \left[s_n + c_n H_n \sum_{j=m+1}^{n} L_j \right]$$

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COMPACT BOOK STORAGE IN A LIBRARY

- **The problem is to find the set of shelf heights and lengths to *minimize* the *total shelving costs***
- **The solution approach is to use a network flow model for a network with**
  - **the set of $(n+1)$ nodes**

$$\mathcal{N} = \{ 0, 1, 2, \ldots, n \}$$

corresponding to the n book heights with

$$1 \leftrightarrow H_1 < H_2 < \ldots < H_n \leftrightarrow n$$

and the starting node with height *0*

COMPACT BOOK STORAGE IN A LIBRARY

- directed arcs (i, j) only if $j > i$ resulting in a

  total of $\dfrac{n(n+1)}{2}$ arcs

- "distance" d_{ij} on each arc given by

$$d_{ij} = \begin{cases} s_j + c_j H_j \displaystyle\sum_{k=i+1}^{j} L_k & \text{if} \quad j > i \\ \infty & \text{otherwise} \end{cases}$$

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COMPACT BOOK STORAGE IN A LIBRARY

- For this network, we solve the shortest route problem for the specified "distances" d_{ij}
- Suppose that for a problem with $n = 17$, we determine the optimal trajectory to be

$$\{ (0,7), (7,9), (9,15), (15,17) \}$$

the interpretation of this solution is :

COMPACT BOOK STORAGE IN A LIBRARY

- store all the books of height $\leq H_7$ on the shelf of height H_7
- store all the books of height $\leq H_9$ but $> H_7$ on the shelf of height H_9
- store all the books of height $\leq H_{15}$ but $> H_9$ on the shelf of height H_{15}
- store all the books of height $\leq H_{17}$ but $> H_{15}$ on the shelf of height H_{17}